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Structure-preserving algorithms for the relativistic Vlasov–Fokker–Planck–Maxwell system

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Summary

✓ Mathematics and physics are two sides of the same coin, even in discrete level

✓ A charge-momentum-energy-conserving finite-difference scheme has been developed for the relativistic Vlasov–Maxwell system

✓ A mass-momentum-energy-conserving finite-difference scheme has been developed for the relativistic Fokker–Planck operator

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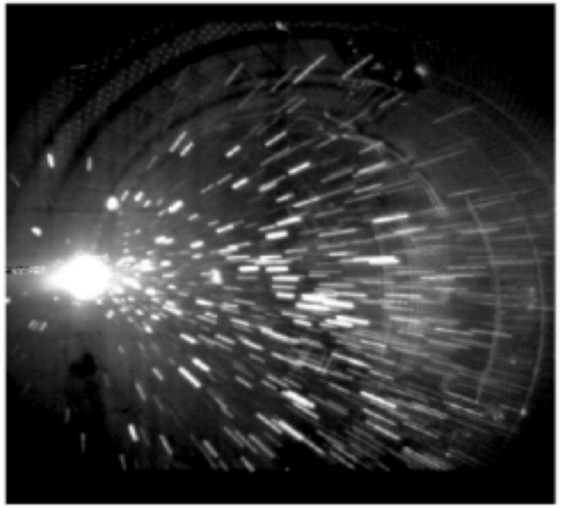
Relativistic kinetic electrons are important to discuss the fusion plasmas

- ✓ RE should be avoided by collision to protect plasma facing components
- ✓ Drag heating is a dominant process in ignition-scale inertial confinement fusion
- ✓ "Numerical heating" can degrade the reliability of kinetic simulations



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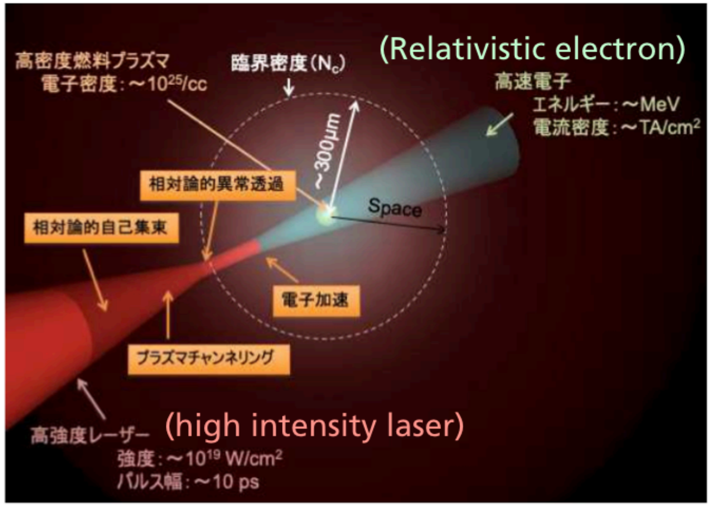

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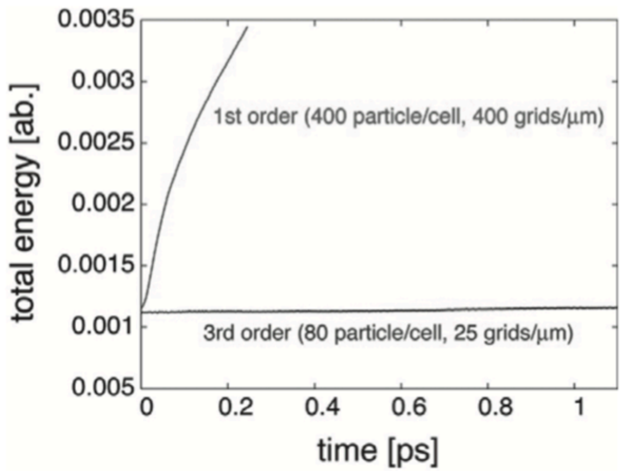

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Relativistic Vlasov–Maxwell

What's the relativistic Vlasov–Maxwell system?

Relativistic Vlasov equation: $$\frac{\partial f}{\partial t} + \frac{\mathbf{u}}{\gamma} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m} \left(\mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{\gamma c} \right) \cdot \frac{\partial f}{\partial \mathbf{u}} = 0$$

Maxwell's equations: $$\mathrm{rot}\ \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \quad \mathrm{rot}\ \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

The product rule, integration-by-parts, and commutative law are required for

- ✓ The Gauss's law
- ✓ The solenoidal constraint of magnetic field
- ✓ The conservation laws of charge, momentum, and energy

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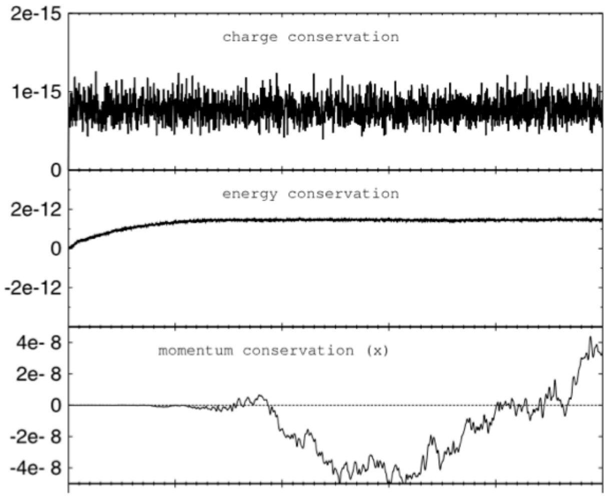
Numerical heating has been a "nightmare" for kinetic plasma scientists

- ✓ Particle-in-cell (PIC) cannot conserve momentum and energy simultaneously
- ✓ Exactly conservative Vlasov scheme was proposed only for periodic geometries
- ✓ Charge- and L2-conserving finite-difference scheme for gyrokinetic simulation



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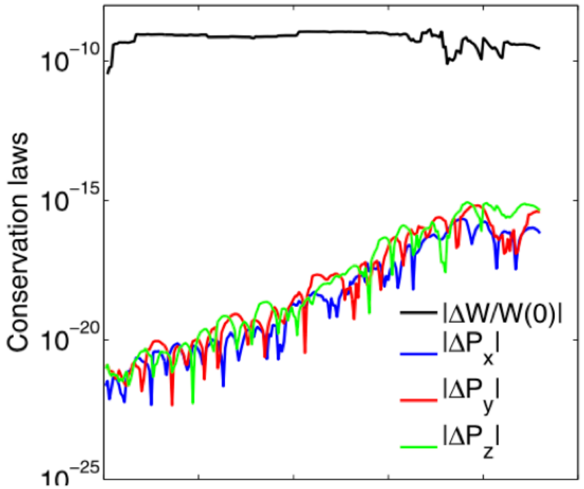

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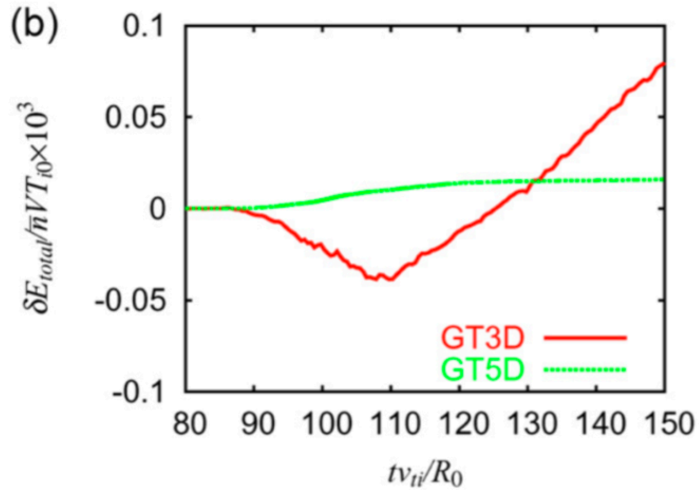

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Concept of the structure-preserving discretization

✓ The velocity should be described as derivative of the Lorentz factor

$$\frac{\partial f}{\partial t}+\frac{\mathbf{u}}{\gamma}\cdot\frac{\partial f}{\partial \mathbf{x}}+\frac{q}{m}\left(\mathbf{E}+\frac{\mathbf{u}\times\mathbf{B}}{\gamma c}\right)\cdot\frac{\partial f}{\partial \mathbf{u}}=0 \;\Rightarrow\; \frac{\partial f}{\partial t}+\frac{\partial}{\partial \mathbf{x}}\cdot\left(\boxed{\frac{\partial \gamma}{\partial \mathbf{u}}}c^2 f\right)+\frac{\partial}{\partial \mathbf{u}}\cdot\left[\frac{q}{m}\left\{\mathbf{E}+\boxed{\frac{\partial \gamma}{\partial \mathbf{u}}}c\times\mathbf{B}\right\}f\right]=0$$

$$\mathbf{J}=q\iiint\frac{\mathbf{u}}{\gamma}f\ \mathrm{d}V \;\Rightarrow\; \mathbf{J}=q\iiint\boxed{\frac{\partial \gamma}{\partial \mathbf{u}}}c^2 f\ \mathrm{d}V$$

✓ Implicit midpoint rule for the momentum-energy conservation

$$\frac{1}{c}\frac{\mathbf{E}^{n+1}-\mathbf{E}^{n}}{\Delta t}+\frac{4\pi}{c}\mathbf{J}^{n+\frac{1}{2}}=\nabla\times\mathbf{B}^{n+\frac{1}{2}},\quad \frac{1}{c}\frac{\mathbf{B}^{n+1}-\mathbf{B}^{n}}{\Delta t}=-\nabla\times\mathbf{E}^{n+\frac{1}{2}},$$

$$\frac{\left(|\mathbf{E}|^2+|\mathbf{B}|^2\right)^{n+1}-\left(|\mathbf{E}|^2+|\mathbf{B}|^2\right)^{n}}{8\pi\Delta t}+\frac{c}{4\pi}\nabla\cdot\left(\mathbf{E}^{\boxed{n+\frac{1}{2}}}\times\mathbf{B}^{\boxed{n+\frac{1}{2}}}\right)=-\mathbf{E}^{\boxed{n+\frac{1}{2}}}\cdot\mathbf{J}^{\boxed{n+\frac{1}{2}}},\qquad \cancel{-(\mathbf{E}\cdot\mathbf{J})^{n+\frac{1}{2}}}$$

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Structure-preserving discretization for the relativistic Vlasov equation

$$
\begin{aligned}
&\frac{\delta}{\delta t}[f^{n+\frac{1}{2},i_1,i_2,i_3,j_1,j_2,j_3}] + \frac{u_x^{j_1}}{\boxed{\gamma^{\tilde{j_1},j_2,j_3}}}\frac{\delta}{\delta x}[f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}] + \frac{u_y^{j_2}}{\boxed{\gamma^{j_1,\tilde{j_2},j_3}}}\frac{\delta}{\delta y}[f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}] + \frac{u_z^{j_3}}{\boxed{\gamma^{j_1,j_2,\tilde{j_3}}}}\frac{\delta}{\delta z}[f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}] \\
&+\frac{qE_x^{\hat{n},i_1,i_2,i_3}}{m}\frac{\delta}{\delta u_x}[f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}] + \frac{qB_z^{\hat{n},i_1,i_2,i_3}u_y^{j_2}}{mc}\frac{\delta}{\delta u_x}\left[\frac{f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}}{\boxed{\gamma^{j_1,\tilde{j_2},j_3}}}\right] - \frac{qB_y^{\hat{n},i_1,i_2,i_3}u_z^{j_3}}{mc}\frac{\delta}{\delta u_x}\left[\frac{f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}}{\boxed{\gamma^{j_1,j_2,\tilde{j_3}}}}\right] \\
&+\frac{qE_y^{\hat{n},i_1,i_2,i_3}}{m}\frac{\delta}{\delta u_y}[f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}] + \frac{qB_x^{\hat{n},i_1,i_2,i_3}u_z^{j_3}}{mc}\frac{\delta}{\delta u_y}\left[\frac{f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}}{\boxed{\gamma^{j_1,j_2,\tilde{j_3}}}}\right] - \frac{qB_z^{\hat{n},i_1,i_2,i_3}u_x^{j_1}}{mc}\frac{\delta}{\delta u_y}\left[\frac{f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}}{\boxed{\gamma^{\tilde{j_1},j_2,j_3}}}\right] \\
&+\frac{qE_z^{\hat{n},i_1,i_2,i_3}}{m}\frac{\delta}{\delta u_z}[f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}] + \frac{qB_y^{\hat{n},i_1,i_2,i_3}u_x^{j_1}}{mc}\frac{\delta}{\delta u_z}\left[\frac{f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}}{\boxed{\gamma^{\tilde{j_1},j_2,j_3}}}\right] - \frac{qB_x^{\hat{n},i_1,i_2,i_3}u_y^{j_2}}{mc}\frac{\delta}{\delta u_z}\left[\frac{f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}}{\boxed{\gamma^{j_1,\tilde{j_2},j_3}}}\right] = 0,
\end{aligned}
$$

$$
\frac{\delta}{\delta t}[F^{n+\frac{1}{2}}] = \frac{F^{n+1}-F^{n}}{\Delta t},\quad \frac{\delta}{\delta x}[F^{i_1}] = \frac{F^{i_1+1}-F^{i_1-1}}{2\Delta x},\quad \frac{\delta}{\delta y}[F^{i_2}] = \frac{F^{i_2+1}-F^{i_2-1}}{2\Delta y},\quad \frac{\delta}{\delta z}[F^{i_3}] = \frac{F^{i_3+1}-F^{i_3-1}}{2\Delta z},
$$

$$
\frac{\delta}{\delta u_x}[F^{j_1}] = \frac{F^{j_1+1}-F^{j_1-1}}{2\Delta u_x},\quad \frac{\delta}{\delta u_y}[F^{j_2}] = \frac{F^{j_2+1}-F^{j_2-1}}{2\Delta u_y},\quad \frac{\delta}{\delta u_z}[F^{j_3}] = \frac{F^{j_3+1}-F^{j_3-1}}{2\Delta u_z},
$$

$$
F^{\hat{n}} = \frac{F^{n+1}+F^{n}}{2},\quad \boxed{F^{\tilde{j_1}} = \frac{F^{j_1+1}+F^{j_1-1}}{2},\quad F^{\tilde{j_2}} = \frac{F^{j_2+1}+F^{j_2-1}}{2},\quad F^{\tilde{j_3}} = \frac{F^{j_3+1}+F^{j_3-1}}{2},}
$$

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Structure-preserving discretization for the Maxwell's equations

$$\frac{\delta}{\delta y}[B_z^{\hat{n},i_1,i_2,i_3}] - \frac{\delta}{\delta z}[B_y^{\hat{n},i_1,i_2,i_3}] = \frac{4\pi}{c} J_x^{\hat{n},i_1,i_2,i_3} + \frac{1}{c}\frac{\delta}{\delta t}[E_x^{n+\frac{1}{2},i_1,i_2,i_3}], \qquad \frac{\delta}{\delta y}[E_z^{\hat{n},i_1,i_2,i_3}] - \frac{\delta}{\delta z}[E_y^{\hat{n},i_1,i_2,i_3}] = -\frac{1}{c}\frac{\delta}{\delta t}[B_x^{n+\frac{1}{2},i_1,i_2,i_3}],$$

$$\frac{\delta}{\delta z}[B_x^{\hat{n},i_1,i_2,i_3}] - \frac{\delta}{\delta x}[B_z^{\hat{n},i_1,i_2,i_3}] = \frac{4\pi}{c} J_y^{\hat{n},i_1,i_2,i_3} + \frac{1}{c}\frac{\delta}{\delta t}[E_y^{n+\frac{1}{2},i_1,i_2,i_3}], \qquad \frac{\delta}{\delta z}[E_x^{\hat{n},i_1,i_2,i_3}] - \frac{\delta}{\delta x}[E_z^{\hat{n},i_1,i_2,i_3}] = -\frac{1}{c}\frac{\delta}{\delta t}[B_y^{n+\frac{1}{2},i_1,i_2,i_3}],$$

$$\frac{\delta}{\delta x}[B_y^{\hat{n},i_1,i_2,i_3}] - \frac{\delta}{\delta y}[B_x^{\hat{n},i_1,i_2,i_3}] = \frac{4\pi}{c} J_z^{\hat{n},i_1,i_2,i_3} + \frac{1}{c}\frac{\delta}{\delta t}[E_z^{n+\frac{1}{2},i_1,i_2,i_3}], \qquad \frac{\delta}{\delta x}[E_y^{\hat{n},i_1,i_2,i_3}] - \frac{\delta}{\delta y}[E_x^{\hat{n},i_1,i_2,i_3}] = -\frac{1}{c}\frac{\delta}{\delta t}[B_z^{n+\frac{1}{2},i_1,i_2,i_3}],$$

$$\rho^{n,i_1,i_2,i_3} = q \sum_{j_1,j_2,j_3} f^{n,i_1,i_2,i_3,j_1,j_2,j_3} \Delta V, \qquad J_x^{n,i_1,i_2,i_3} = q \sum_{j_1,j_2,j_3} \frac{u_x^{j_1}}{\boxed{\gamma^{\tilde{j_1},j_2,j_3}}} f^{n,i_1,i_2,i_3,j_1,j_2,j_3} \Delta V,$$

$$J_y^{n,i_1,i_2,i_3} = q \sum_{j_1,j_2,j_3} \frac{u_y^{j_2}}{\boxed{\gamma^{j_1,\tilde{j_2},j_3}}} f^{n,i_1,i_2,i_3,j_1,j_2,j_3} \Delta V, \qquad J_z^{n,i_1,i_2,i_3} = q \sum_{j_1,j_2,j_3} \frac{u_z^{j_3}}{\boxed{\gamma^{j_1,j_2,\tilde{j_3}}}} f^{n,i_1,i_2,i_3,j_1,j_2,j_3} \Delta V,$$

$$\sum_{j_1,j_2,j_3} = \sum_{j_1}\sum_{j_2}\sum_{j_3}, \qquad \Delta V = \Delta u_x \Delta u_y \Delta u_z,$$

Product rules, integration-by-parts and commutative laws in discrete form

Product rules:

$$\frac{\delta}{\delta u_x}[F^{j_1}]G^{j_1} + F^{j_1}\frac{\delta}{\delta u_x}[G^{j_1}] = \frac{\boxed{F^{j_1+1}G^{j_1}} + \boxed{F^{j_1}G^{j_1+1}} - \boxed{F^{j_1}G^{j_1-1}} - \boxed{F^{j_1-1}G^{j_1}}}{2\Delta u_x} \equiv \frac{\mathrm{D}}{\mathrm{D}u_x}[F^{j_1}, G^{j_1}].$$

$$\frac{\delta}{\delta t}\left[F^{n+\frac{1}{2}}\right]\frac{G^{n+1}+G^{n}}{2} + \frac{F^{n+1}+F^{n}}{2}\frac{\delta}{\delta t}\left[G^{n+\frac{1}{2}}\right] = \frac{\delta}{\delta t}\left[(FG)^{n+\frac{1}{2}}\right]$$

Integration-by-parts:

$$\sum_{j_1,j_2,j_3}\frac{\delta}{\delta u_x}[F^{j_1}]G^{j_1}\Delta V = -\sum_{j_1,j_2,j_3}F^{j_1}\frac{\delta}{\delta u_x}[G^{j_1}]\Delta V.$$

$F^1 = F^2 = F^{M_x-1} = F^{M_x} = 0$ **is required!**

Commutative laws:

$$\frac{\delta}{\delta x}\left[\frac{\delta}{\delta y}[F^{i_1,i_2}]\right] = \frac{F^{i_1+1,j_2+1} - F^{i_1+1,j_2-1} - F^{i_1-1,i_2+1} + F^{i_1-1,i_2-1}}{\Delta x\Delta y} = \frac{\delta}{\delta y}\left[\frac{\delta}{\delta x}[F^{i_1,i_2}]\right],$$

$$\frac{\delta}{\delta t}\left[\frac{\delta}{\delta x}[F^{n+\frac{1}{2},i_1}]\right] = \frac{\delta}{\delta x}\left[\frac{\delta}{\delta t}[F^{n+\frac{1}{2},i_1}]\right].$$

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Charge conservation, Gauss's law and solenoidal constraint for magnetic field in discrete form

Charge conservation:

$$\frac{\delta}{\delta t}[\rho^{n+\frac{1}{2},i_1,i_2,i_3}] + \frac{\delta}{\delta x}[J_x^{\hat{n},i_1,i_2,i_3}] + \frac{\delta}{\delta y}[J_y^{\hat{n},i_1,i_2,i_3}] + \frac{\delta}{\delta z}[J_z^{\hat{n},i_1,i_2,i_3}] = 0.$$

Gauss's law:

$$\frac{\delta}{\delta x}[E_x^{n+1,i_1,i_2,i_3}] + \frac{\delta}{\delta y}[E_y^{n+1,i_1,i_2,i_3}] + \frac{\delta}{\delta z}[E_z^{n+1,i_1,i_2,i_3}] = 4\pi\rho^{n+1,i_1,i_2,i_3}$$

$$\text{if} \quad \frac{\delta}{\delta x}[E_x^{0,i_1,i_2,i_3}] + \frac{\delta}{\delta y}[E_y^{0,i_1,i_2,i_3}] + \frac{\delta}{\delta z}[E_z^{0,i_1,i_2,i_3}] = 4\pi\rho^{0,i_1,i_2,i_3}.$$

Solenoidal constraint:

$$\frac{\delta}{\delta x}[B_x^{n+1,i_1,i_2,i_3}] + \frac{\delta}{\delta y}[B_y^{n+1,i_1,i_2,i_3}] + \frac{\delta}{\delta z}[B_z^{n+1,i_1,i_2,i_3}] = 0$$

$$\text{if} \quad \frac{\delta}{\delta x}[B_x^{0,i_1,i_2,i_3}] + \frac{\delta}{\delta y}[B_y^{0,i_1,i_2,i_3}] + \frac{\delta}{\delta z}[B_z^{0,i_1,i_2,i_3}] = 0.$$

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The law of momentum conservation in discrete form

$$\frac{\delta}{\delta t}\left[\sum_{j_1,j_2,j_3} m u_x^{j_1} f^{n+\frac{1}{2},i_1,i_2,i_3,j_1,j_2,j_3}\Delta V\right] + \frac{\delta}{\delta x}\left[\sum_{j_1,j_2,j_3}\frac{m u_x^{j_1} u_x^{j_1}}{\gamma^{\tilde{j}_1,j_2,j_3}} f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}\Delta V\right] + \frac{\delta}{\delta y}\left[\sum_{j_1,j_2,j_3}\frac{m u_x^{j_1} u_y^{j_2}}{\gamma^{j_1,\tilde{j}_2,j_3}} f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}\Delta V\right]$$

$$+\frac{\delta}{\delta z}\left[\sum_{j_1,j_2,j_3}\frac{m u_x^{j_1} u_z^{j_3}}{\gamma^{j_1,j_2,\tilde{j}_3}} f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}\Delta V\right] = \boxed{\rho^{\hat{n},i_1,i_2,i_3} E_x^{\hat{n},i_1,i_2,i_3} + \frac{J_y^{\hat{n},i_1,i_2,i_3} B_z^{\hat{n},i_1,i_2,i_3} - J_z^{\hat{n},i_1,i_2,i_3} B_y^{\hat{n},i_1,i_2,i_3}}{c}.}$$

$$\frac{\delta}{\delta t}\left[\frac{(E_y B_z - E_z B_y)^{n+\frac{1}{2},i_1,i_2,i_3}}{4\pi c}\right] - \frac{1}{8\pi}\frac{\mathrm{D}}{\mathrm{D}x}[E_x^{\hat{n},i_1,i_2,i_3}, E_x^{\hat{n},i_1,i_2,i_3}] + \frac{1}{8\pi}\frac{\mathrm{D}}{\mathrm{D}x}[E_y^{\hat{n},i_1,i_2,i_3}, E_y^{\hat{n},i_1,i_2,i_3}] + \frac{1}{8\pi}\frac{\mathrm{D}}{\mathrm{D}x}[E_z^{\hat{n},i_1,i_2,i_3}, E_z^{\hat{n},i_1,i_2,i_3}]$$

$$-\frac{1}{8\pi}\frac{\mathrm{D}}{\mathrm{D}x}[B_x^{\hat{n},i_1,i_2,i_3}, B_x^{\hat{n},i_1,i_2,i_3}] + \frac{1}{8\pi}\frac{\mathrm{D}}{\mathrm{D}x}[B_y^{\hat{n},i_1,i_2,i_3}, B_y^{\hat{n},i_1,i_2,i_3}] + \frac{1}{8\pi}\frac{\mathrm{D}}{\mathrm{D}x}[B_z^{\hat{n},i_1,i_2,i_3}, B_z^{\hat{n},i_1,i_2,i_3}] - \frac{1}{4\pi}\frac{\mathrm{D}}{\mathrm{D}y}[E_x^{\hat{n},i_1,i_2,i_3}, E_y^{\hat{n},i_1,i_2,i_3}]$$

$$-\frac{1}{4\pi}\frac{\mathrm{D}}{\mathrm{D}y}[B_x^{\hat{n},i_1,i_2,i_3}, B_y^{\hat{n},i_1,i_2,i_3}] - \frac{1}{4\pi}\frac{\mathrm{D}}{\mathrm{D}z}[E_x^{\hat{n},i_1,i_2,i_3}, E_z^{\hat{n},i_1,i_2,i_3}] - \frac{1}{4\pi}\frac{\mathrm{D}}{\mathrm{D}z}[B_x^{\hat{n},i_1,i_2,i_3}, B_z^{\hat{n},i_1,i_2,i_3}]$$

$$= \boxed{-\rho^{\hat{n},i_1,i_2,i_3} E_x^{\hat{n},i_1,i_2,i_3} - \frac{J_y^{\hat{n},i_1,i_2,i_3} B_z^{\hat{n},i_1,i_2,i_3} - J_z^{\hat{n},i_1,i_2,i_3} B_y^{\hat{n},i_1,i_2,i_3}}{c},}$$

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The law of energy conservation in discrete form

$$\frac{\delta}{\delta t}\left[\sum_{j_1,j_2,j_3}\gamma^{j_1,j_2,j_3}mc^2 f^{n+\frac{1}{2},i_1,i_2,i_3,j_1,j_2,j_3}\Delta V\right]+\frac{\delta}{\delta x}\left[\sum_{j_1,j_2,j_3}\frac{mc^2u_x^{j_1}\gamma^{j_1,j_2,j_3}}{\gamma^{\tilde{j}_1,j_2,j_3}}f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}\Delta V\right]$$

$$+\frac{\delta}{\delta y}\left[\sum_{j_1,j_2,j_3}\frac{mc^2u_y^{j_2}\gamma^{j_1,j_2,j_3}}{\gamma^{j_1,\tilde{j}_2,j_3}}f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}\Delta V\right]+\frac{\delta}{\delta z}\left[\sum_{j_1,j_2,j_3}\frac{mc^2u_z^{j_3}\gamma^{j_1,j_2,j_3}}{\gamma^{j_1,j_2,\tilde{j}_3}}f^{\hat{n},i_1,i_2,i_3,j_1,j_2,j_3}\Delta V\right]=\boxed{\mathbf{J}^{\hat{n},i_1,i_2,i_3}\cdot\mathbf{E}^{\hat{n},i_1,i_2,i_3}}$$

$$\left(\because\ \frac{\delta}{\delta u_x}\left[\gamma^{j_1,j_2,j_3}\right]=\frac{1}{c^2}\frac{{u_x}^{j_1+1}+{u_x}^{j_1-1}}{\gamma^{j_1+1,j_2,j_3}+\gamma^{j_1-1,j_2,j_3}}=\frac{1}{c^2}\frac{{u_x}^{j_1}}{\gamma^{\tilde{j}_1,j_2,j_3}}\right)$$

$$\frac{\delta}{\delta t}\left[\frac{(\mathbf{E}^2+\mathbf{B}^2)^{n+\frac{1}{2},i_1,i_2,i_3}}{8\pi}\right]+\frac{c}{4\pi}\frac{\mathrm{D}}{\mathrm{D}x}[E_y^{\hat{n},i_1,i_2,i_3},B_z^{\hat{n},i_1,i_2,i_3}]-\frac{c}{4\pi}\frac{\mathrm{D}}{\mathrm{D}x}[E_z^{\hat{n},i_1,i_2,i_3},B_y^{\hat{n},i_1,i_2,i_3}]+\frac{c}{4\pi}\frac{\mathrm{D}}{\mathrm{D}y}[E_z^{\hat{n},i_1,i_2,i_3},B_x^{\hat{n},i_1,i_2,i_3}]$$

$$-\frac{c}{4\pi}\frac{\mathrm{D}}{\mathrm{D}y}[E_x^{\hat{n},i_1,i_2,i_3},B_z^{\hat{n},i_1,i_2,i_3}]+\frac{c}{4\pi}\frac{\mathrm{D}}{\mathrm{D}z}[E_x^{\hat{n},i_1,i_2,i_3},B_y^{\hat{n},i_1,i_2,i_3}]-\frac{c}{4\pi}\frac{\mathrm{D}}{\mathrm{D}z}[E_y^{\hat{n},i_1,i_2,i_3},B_x^{\hat{n},i_1,i_2,i_3}]=\boxed{-\mathbf{J}^{\hat{n},i_1,i_2,i_3}\cdot\mathbf{E}^{\hat{n},i_1,i_2,i_3}}$$

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How to ensure "magnetic forces do not work"

Simplified Vlasov equation in discrete form: $\frac{\delta f}{\delta t} + \frac{q}{m}\frac{\delta}{\delta \mathbf{u}} \cdot \left[\left(\frac{1}{c}\frac{\delta(\gamma c^2)}{\delta \mathbf{u}} \times \mathbf{B}\right) f\right] = 0$

$$\sum \frac{\delta f}{\delta t}\gamma m c^2 \Delta\mathbf{u} + \sum q\gamma c^2 \frac{\delta}{\delta \mathbf{u}} \cdot \left[\left(\frac{1}{c}\frac{\delta(\gamma c^2)}{\delta \mathbf{u}} \times \mathbf{B}\right) f\right] \Delta\mathbf{u} = 0$$

Perpendicular to $\frac{\delta(\gamma c^2)}{\delta \mathbf{u}}$

$$\sum \frac{\delta f}{\delta t}\gamma m c^2 \Delta\mathbf{u} - \sum q\frac{\delta(\gamma c^2)}{\delta \mathbf{u}} \cdot \left[\boxed{\left(\frac{1}{c}\frac{\delta(\gamma c^2)}{\delta \mathbf{u}} \times \mathbf{B}\right)} f\right] \Delta\mathbf{u} = 0$$

$$\therefore \sum \frac{\delta f}{\delta t}\gamma m c^2 \Delta\mathbf{u} = 0$$

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Electromagnetic experiment via relativistic Weibel instability

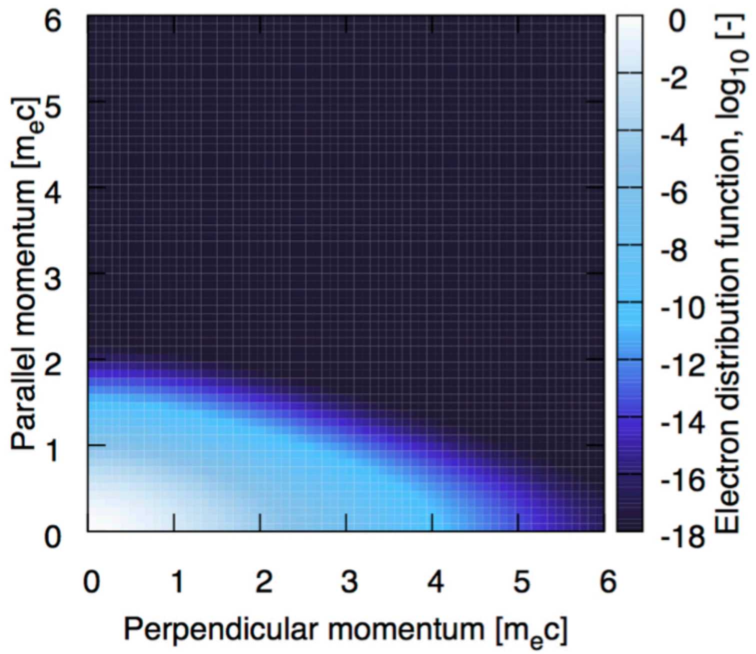
Hydrogen plasma:

$$q_\mathrm{i}/q_\mathrm{e} = -1, \qquad m_\mathrm{i}/m_\mathrm{e} = 1836.$$

Relativistic bi-Maxwell distribution:

$$f_\mathrm{init} \propto \exp\left[-\alpha_\perp\left(\gamma-\gamma_\parallel\right)-\alpha_\parallel\gamma_\parallel\right],$$
$$\alpha_{\mathrm{e},\parallel} = 30, \qquad \alpha_{\mathrm{e},\perp} = \alpha_\mathrm{i} = 5.$$

Computational grids: 128 × 128 × 128 × 128



Most unstable mode: $kc/\omega_\mathrm{pe} \simeq 0.92, \qquad \Gamma/\omega_\mathrm{pe} \simeq 0.144,$

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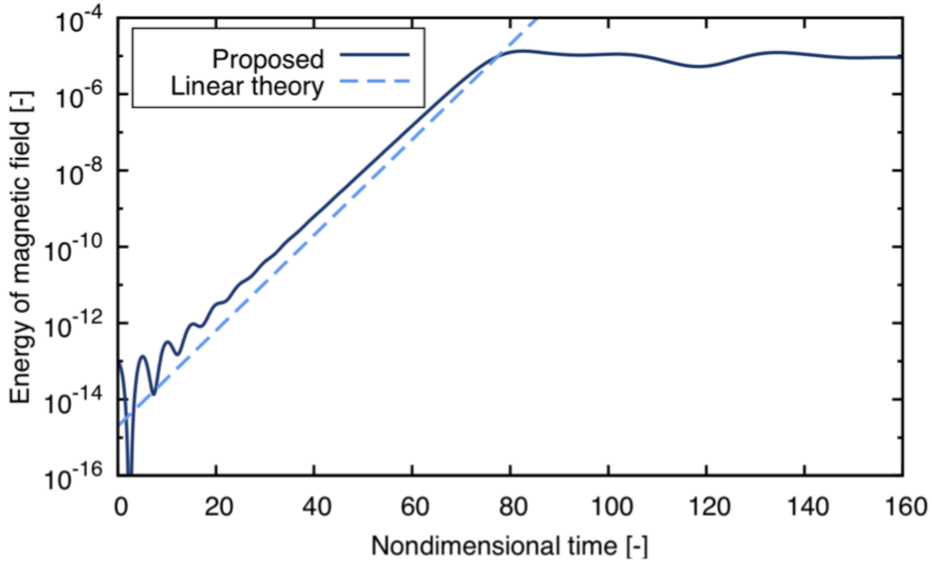
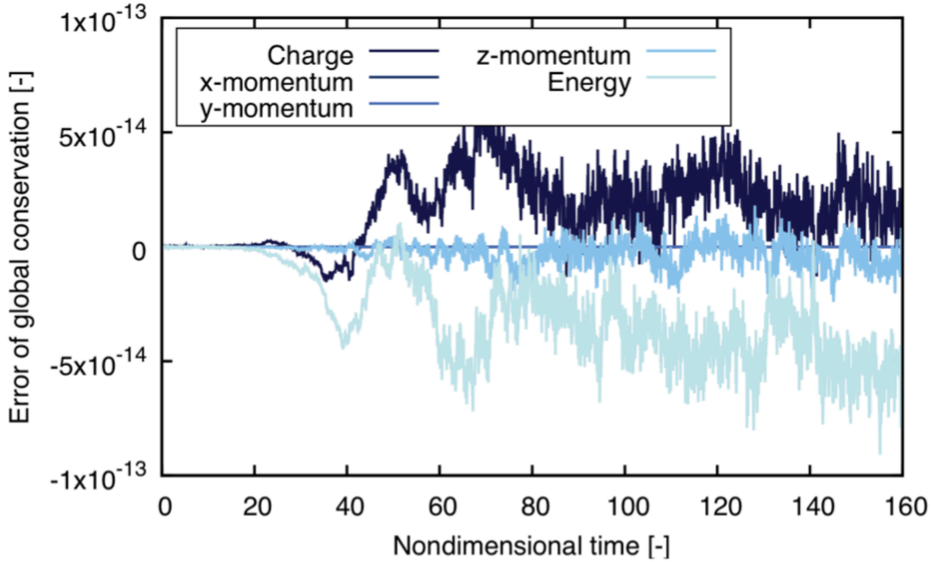
Charge-momentum-energy-conserving relativistic Vlasov simulation has been demonstrated

Takashi Shiroto, Naofumi Ohnishi and Yasuhiko Sentoku, J. Comput. Phys. 379, 32 (2019).

- ✓ Linear theory is reproduced well in the numerical experiment
- ✓ The conservation laws are strictly maintained even in the nonlinear phase
- ✓ Conservative Vlasov–Fokker–Planck–Maxwell simulation is under construction





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Conclusion of this section

Conservative relativistic Vlasov–Maxwell simulation has been demonstrated.

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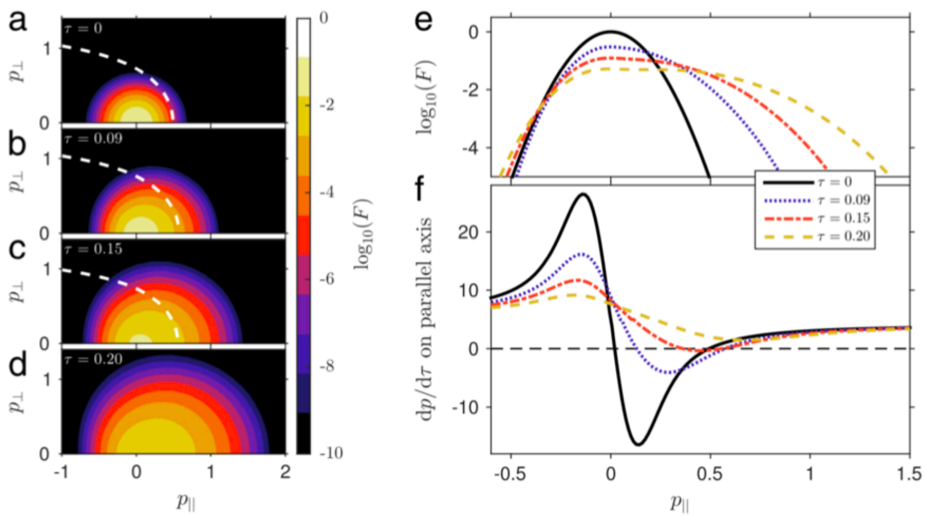
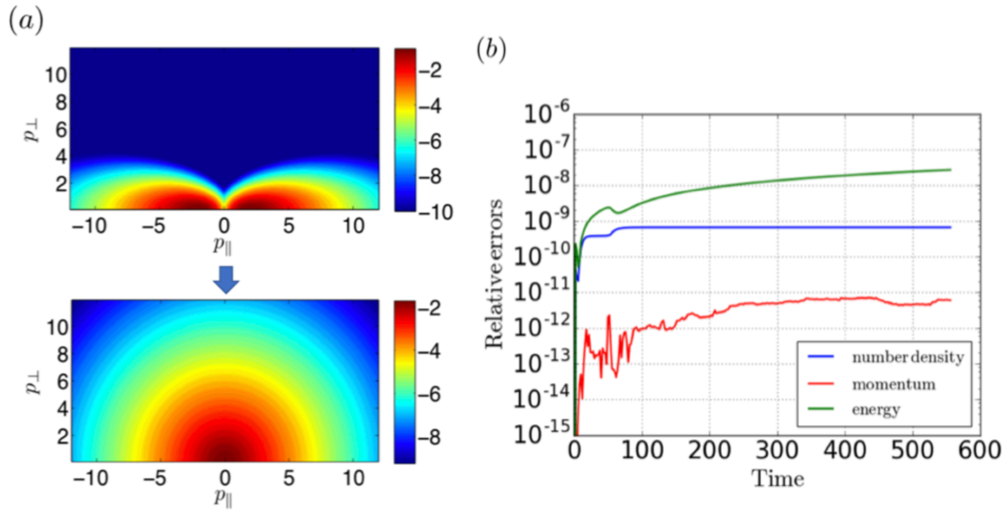
Relativistic Fokker–Planck

Relativistic Fokker–Planck simulation becomes more and more important for the runaway electrons

- ✓ NORSE code enables to simulate the electron slide-away by O(N)
- ✓ Conservation laws are enforced by "nonlinear constraints"
- ✓ Structure-preserving scheme has not been proposed



A. Stahl et al., Comput. Phys. Commun. (2017).



D. Daniel et al., arXiv (2019).

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Mass-momentum-energy-conserving scheme for relativistic Fokker–Planck operator had not been proposed

- ✓ Conservative nonlinear Fokker–Planck schemes were proposed in 1980s
- ✓ Conservative finite-element scheme in the non-relativistic regime
- ✓ Lorentz factor is difficult to be expressed with finite order of accuracy

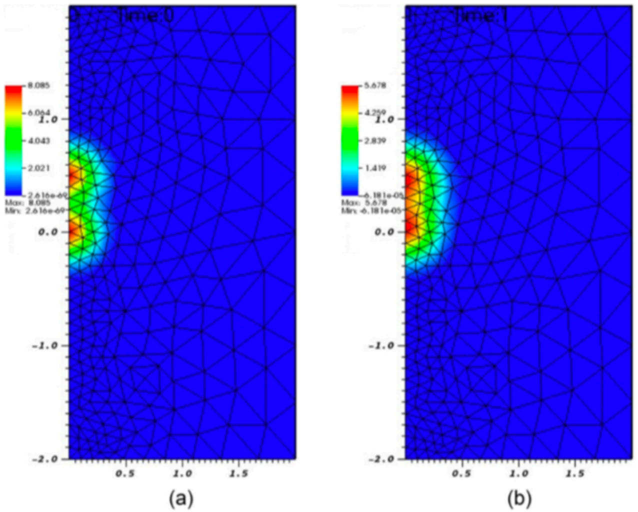
Conservative scheme by "Log-form"

$$\frac{\partial f}{\partial t} = (1/\sqrt{\varepsilon})\frac{\partial}{\partial \varepsilon}\int_0^{\varepsilon_0}\left(f'\frac{\partial f}{\partial \varepsilon} - f\frac{\partial f'}{\partial \varepsilon'}\right) g(\varepsilon, \varepsilon')\, d\varepsilon'$$

↕

$$\frac{\partial f}{\partial t} = \frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial \varepsilon}\int_0^{\varepsilon_0}\left(\frac{\partial}{\partial \varepsilon}\ln f - \frac{\partial}{\partial \varepsilon'}\ln f'\right) ff'g(\varepsilon, \varepsilon')\, d\varepsilon'.$$

Yu. A. Berezin et al., J. Comput. Phys. (1987).



E. Hirvijoki and M.F. Adams, Phys. Plasmas (2017).

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What's the relativistic Landau–Fokker–Planck equation?

The relativistic Landau–Fokker–Planck (RLFP) equation:

$$\frac{\partial f_s}{\partial t} = \frac{\Gamma_{s/s'}}{2}\frac{\partial}{\partial \mathbf{u}} \cdot \int \mathsf{U}(\mathbf{u},\mathbf{u}') \cdot \left(f_{s'}\frac{\partial f_s}{\partial \mathbf{u}} - \frac{m_s}{m_{s'}}\frac{\partial f_{s'}}{\partial \mathbf{u}'} f_s \right) \mathrm{d}\mathbf{u}', \quad \Gamma_{s/s'} = \frac{{q_s}^2 {q_{s'}}^2 \log \Lambda_{s/s'}}{4\pi {\varepsilon_0}^2 {m_s}^2},$$

The Beliaev–Budker kernel:

$$\mathsf{U}(\mathbf{u},\mathbf{u}') = \frac{r^2}{\gamma\gamma' w^3}\left[w^2 \mathsf{I} - \mathbf{u}\mathbf{u} - \mathbf{u}'\mathbf{u}' + r\left(\mathbf{u}\mathbf{u}' + \mathbf{u}'\mathbf{u}\right)\right],$$

$$\gamma = \sqrt{1 + \mathbf{u}\cdot\mathbf{u}/c^2}, \quad \mathbf{v} = \mathbf{u}/\gamma, \quad r = \gamma\gamma' - \mathbf{u}\cdot\mathbf{u}'/c^2, \quad w = c\sqrt{r^2 - 1},$$

$$\boxed{\mathsf{U}(\mathbf{u},\mathbf{u}') = \mathsf{U}(\mathbf{u}',\mathbf{u}), \quad \mathsf{U}(\mathbf{u},\mathbf{u}')\cdot\mathbf{v} = \mathsf{U}(\mathbf{u}',\mathbf{u})\cdot\mathbf{v}',}$$

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Integration-by-parts is an important player to derive the conservation laws in weak formulation

Weak formulation of the RLFP equation:

$$\int \frac{\partial f_s}{\partial t} m_s \phi \, \mathrm{d}\mathbf{u} = \frac{\Gamma_{s/s'} {m_s}^2}{2} \int \phi \frac{\partial}{\partial \mathbf{u}} \cdot \int \mathsf{U}(\mathbf{u}, \mathbf{u}') \cdot \left(\frac{1}{m_s} f_{s'} \frac{\partial f_s}{\partial \mathbf{u}} - \frac{1}{m_{s'}} \frac{\partial f_{s'}}{\partial \mathbf{u}'} f_s \right) \mathrm{d}\mathbf{u}' \mathrm{d}\mathbf{u}$$

$$= -\frac{\Gamma_{s/s'} {m_s}^2}{2} \iint \mathsf{U}(\mathbf{u}, \mathbf{u}') \cdot \frac{\partial \phi}{\partial \mathbf{u}} \cdot \left(\frac{1}{m_s} f_{s'} \frac{\partial f_s}{\partial \mathbf{u}} - \frac{1}{m_{s'}} \frac{\partial f_{s'}}{\partial \mathbf{u}'} f_s \right) \mathrm{d}\mathbf{u}' \mathrm{d}\mathbf{u},$$

Integration-by-parts

$$\mathsf{U}(\mathbf{u}, \mathbf{u}') = \mathsf{U}(\mathbf{u}', \mathbf{u}),$$
$$\mathsf{U}(\mathbf{u}, \mathbf{u}') \cdot \mathbf{v} = \mathsf{U}(\mathbf{u}', \mathbf{u}) \cdot \mathbf{v}',$$

Mass-momentum-energy conservation: $\phi = 1, \ \mathbf{u}, \ \gamma c^2$

$$\int \frac{\partial f_s}{\partial t} m_s \phi \, \mathrm{d}\mathbf{u} + \int \frac{\partial f_{s'}}{\partial t} m_{s'} \phi \, \mathrm{d}\mathbf{u}'$$

$$= -\frac{\Gamma_{s/s'} {m_s}^2}{2} \iint \left[\mathsf{U}(\mathbf{u}, \mathbf{u}') \cdot \frac{\partial \phi}{\partial \mathbf{u}} - \mathsf{U}(\mathbf{u}', \mathbf{u}) \cdot \frac{\partial \phi}{\partial \mathbf{u}'} \right] \cdot \left(\frac{1}{m_s} f_{s'} \frac{\partial f_s}{\partial \mathbf{u}} - \frac{1}{m_{s'}} \frac{\partial f_{s'}}{\partial \mathbf{u}'} f_s \right) \mathrm{d}\mathbf{u}' \mathrm{d}\mathbf{u},$$

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Summation-by-parts can destroy the energy conservation

Conservation laws in discrete form:

$$\frac{\delta F_{\mathbf{i}}}{\delta u_x} \equiv \frac{F_{i+1,j,k} - F_{i-1,j,k}}{2\Delta y_x}$$

$$\sum_{\mathbf{i}} \frac{\partial f_{s,\mathbf{i}}}{\partial t} m_s \phi_{\mathbf{i}}\ \Delta \mathbf{u} + \sum_{\mathbf{j}} \frac{\partial f_{s',\mathbf{j}}}{\partial t} m_{s'} \phi_{\mathbf{j}}\ \Delta \mathbf{u}'$$

$$= -\frac{\Gamma_{s/s'} {m_s}^2}{2} \sum_{\mathbf{i},\mathbf{j}} \left[\mathsf{U}(\mathbf{u}_{\mathbf{i}}, \mathbf{u}'_{\mathbf{j}}) \cdot \frac{\delta \phi_{\mathbf{i}}}{\delta \mathbf{u}} - \mathsf{U}(\mathbf{u}'_{\mathbf{j}}, \mathbf{u}_{\mathbf{i}}) \cdot \frac{\delta \phi_{\mathbf{j}}}{\delta \mathbf{u}'} \right] \cdot \left(\frac{1}{m_s} f_{s',\mathbf{j}} \frac{\delta f_{s,\mathbf{i}}}{\delta \mathbf{u}} - \frac{1}{m_{s'}} \frac{\delta f_{s',\mathbf{j}}}{\delta \mathbf{u}'} f_{s,\mathbf{i}} \right) \Delta \mathbf{u}' \Delta \mathbf{u},$$

The law of energy conservation is violated in a conventional way:

$$\frac{\delta \mathbf{u}_{\mathbf{i}}}{\delta \mathbf{u}} = \mathsf{I}, \quad \frac{\delta(\gamma_{\mathbf{i}} c^2)}{\delta \mathbf{u}} \equiv \tilde{\mathbf{v}}_{\mathbf{i}} \neq \mathbf{v}_{\mathbf{i}} = \frac{\mathbf{u}_{\mathbf{i}}}{\gamma_{\mathbf{i}}}$$

$$\mathsf{U}(\mathbf{u}_{\mathbf{i}}, \mathbf{u}'_{\mathbf{j}}) = \mathsf{U}(\mathbf{u}'_{\mathbf{j}}, \mathbf{u}_{\mathbf{i}}),$$

$$\mathsf{U}(\mathbf{u}_{\mathbf{i}}, \mathbf{u}'_{\mathbf{j}}) \cdot \mathbf{v}_{\mathbf{i}} = \mathsf{U}(\mathbf{u}'_{\mathbf{j}}, \mathbf{u}_{\mathbf{i}}) \cdot \mathbf{v}'_{\mathbf{j}},$$

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Strategy for structure-preserving scheme

- ✓ The arguments of collision kernel should be velocity, not the momentum per unit mass
- ✓ Energy conservation is preserved if the kernel is composed by the velocity including truncation errors

$$\boxed{\mathsf{U}(\mathbf{v},\mathbf{v}')} = \frac{r^2}{\gamma\gamma' w^3}\left[w^2\mathsf{I} - \gamma^2\mathbf{v}\mathbf{v} - \gamma'^2\mathbf{v}'\mathbf{v}' + r\gamma\gamma'\left(\mathbf{v}\mathbf{v}' + \mathbf{v}'\mathbf{v}\right)\right],$$

$$\gamma = \frac{1}{\sqrt{1-\mathbf{v}\cdot\mathbf{v}/c^2}} \quad r = \gamma\gamma'\left(1-\mathbf{v}\cdot\mathbf{v}'/c^2\right), \quad w = c\sqrt{r^2-1},$$

$$\mathsf{U}(\mathbf{v},\mathbf{v}') = \mathsf{U}(\mathbf{v}',\mathbf{v}), \quad \mathsf{U}(\mathbf{v},\mathbf{v}')\cdot\mathbf{v} = \mathsf{U}(\mathbf{v}',\mathbf{v})\cdot\mathbf{v}',$$

$$\boxed{\mathsf{U}(\tilde{\mathbf{v}},\tilde{\mathbf{v}}') = \mathsf{U}(\tilde{\mathbf{v}}',\tilde{\mathbf{v}}), \quad \mathsf{U}(\tilde{\mathbf{v}},\tilde{\mathbf{v}}')\cdot\tilde{\mathbf{v}} = \mathsf{U}(\tilde{\mathbf{v}}',\tilde{\mathbf{v}})\cdot\tilde{\mathbf{v}}',}$$

$$\frac{\delta\mathbf{u_i}}{\delta\mathbf{u}} = \mathsf{I}, \quad \frac{\delta(\gamma_\mathbf{i}c^2)}{\delta\mathbf{u}} \equiv \boxed{\tilde{\mathbf{v}}_\mathbf{i}} \neq \mathbf{v_i} = \frac{\mathbf{u_i}}{\gamma_\mathbf{i}}$$

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The proposed and conventional schemes are examined by a relativistic thermal-equilibration test

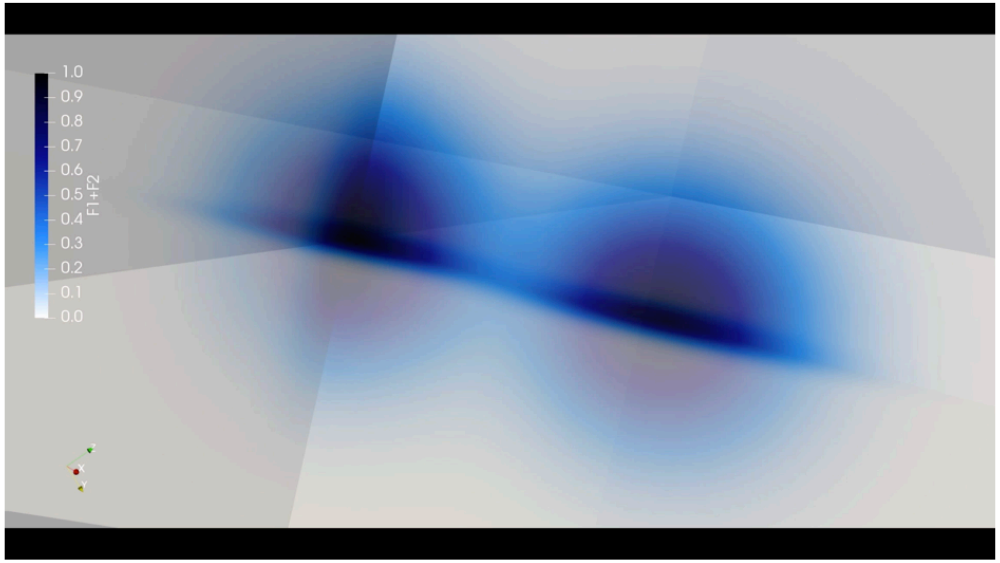
- ✓ Multi-species equilibration through collisional process is calculated
- ✓ Each distribution function is initialized with shifted Maxwell–Jüttner distribution
- ✓ Equilibrium is obtained with expected time-scale

Experimental conditions for thermal-equilibration test:

$$m_s = m_{s'}, \quad q_s = -q_{s'}, \quad f_s(\mathbf{u}) = \exp\left(-\frac{\gamma - 1}{\theta}\right),$$

$$f_{s'}(\mathbf{u}') = \frac{1}{\gamma_0}\exp\left(-\frac{\gamma'\gamma_0 - \gamma_0 \mathbf{v_0}\cdot\mathbf{u}'/c^2 - 1}{\theta}\right),$$

$$\theta = 0.01, \quad \gamma_0 \simeq 1.07, \quad \frac{\Gamma}{m^2}\Delta t = \frac{1}{80},$$



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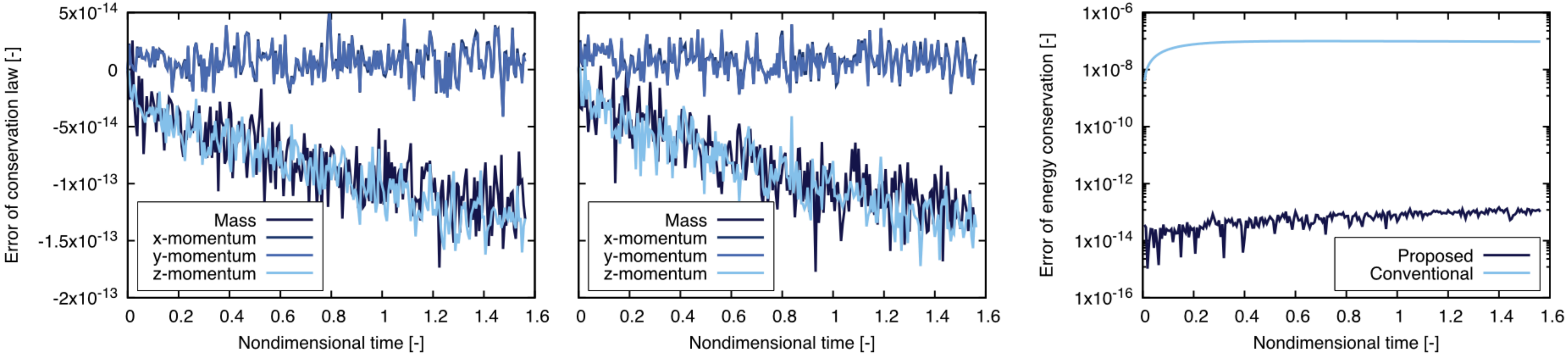
Fully-conservative RLFP scheme has been demonstrated

Takashi Shiroto and Yasuhiko Sentoku, Phys. Rev. E 99, 053309 (2019)

- ✓ Both scheme maintains the mass-momentum conservation exactly
- ✓ Structure-preserving scheme is the only way to preserve the energy conservation
- ✓ One node-month per single spatial point due to the O(N²) computational cost

Proposed scheme | Conventional scheme



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Conclusion of this section

Conservative relativistic Fokker–Planck simulation has been demonstrated.

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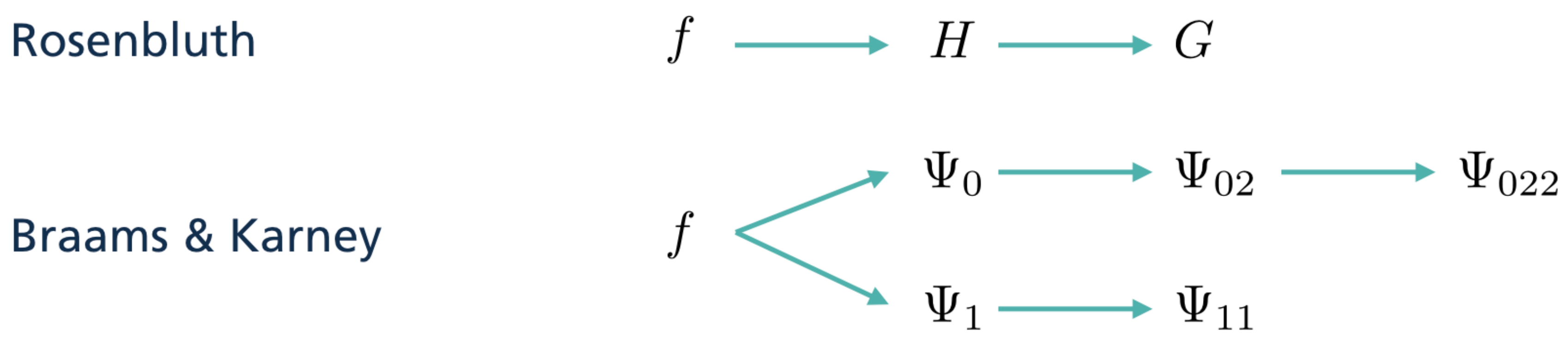
Future work

- ✓ Conservative Rosenbluth scheme is being developed with the discontinuous Galerkin method
- ✓ How to discuss the relativistic (Braams & Karney) extension?



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Summary

- ✓ Mathematics and physics are two sides of the same coin, even in discrete level
- ✓ A charge-momentum-energy-conserving finite-difference scheme has been developed for the relativistic Vlasov–Maxwell system
- ✓ A mass-momentum-energy-conserving finite-difference scheme has been developed for the relativistic Fokker–Planck operator

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