Theory for runaway electron dominated equilibria: status and the way forward

Joint Runaway Electron Modelling (REM) and JET SPI Analysis meeting

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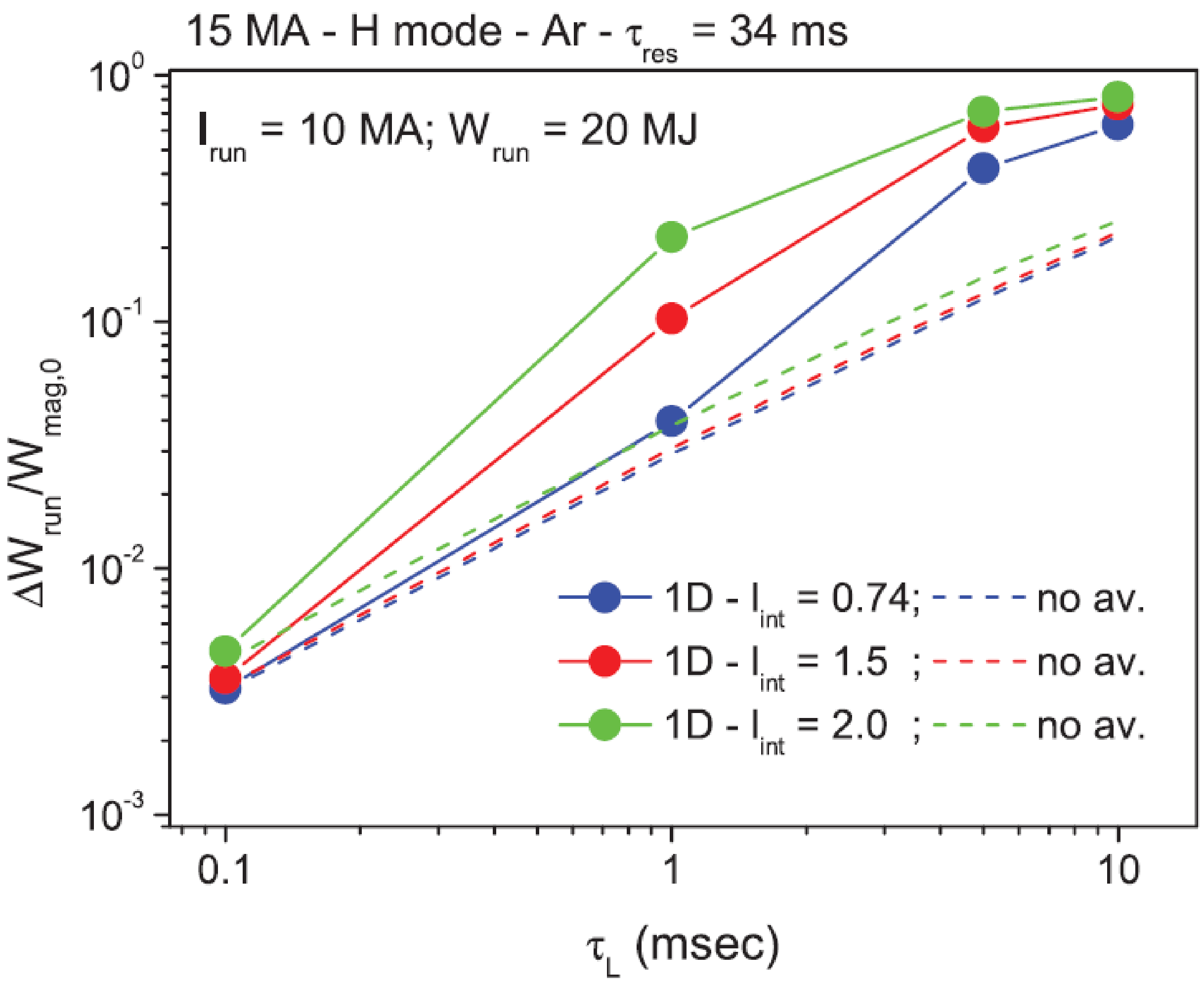
Content

- The runaway electron dominated equilibria
- Ingredients for an equilibrium theory and its status
- The Grad-Shafranov-like equilibrium for RE drift surfaces
- Numerical results: equilibria solutions
- The way forward?
- Conclusion

Runaway electron loss

- REs are the most severe threat to ITER operation.
- Significant **energy conversion** could occur as the RE scraps off.
- The timescale of the RE loss determines **the fraction of the magnetic energy conversion to the kinetic energy**.
- The RE beam **stability study** is important!



J.R. Martín-Solís et al 2017 Nucl. Fusion 57 066025

Runaway electron equilibria

- However, **stability analysis** depends on correct **equilibria description**…… **Accurate RE equilibria theory needed**!

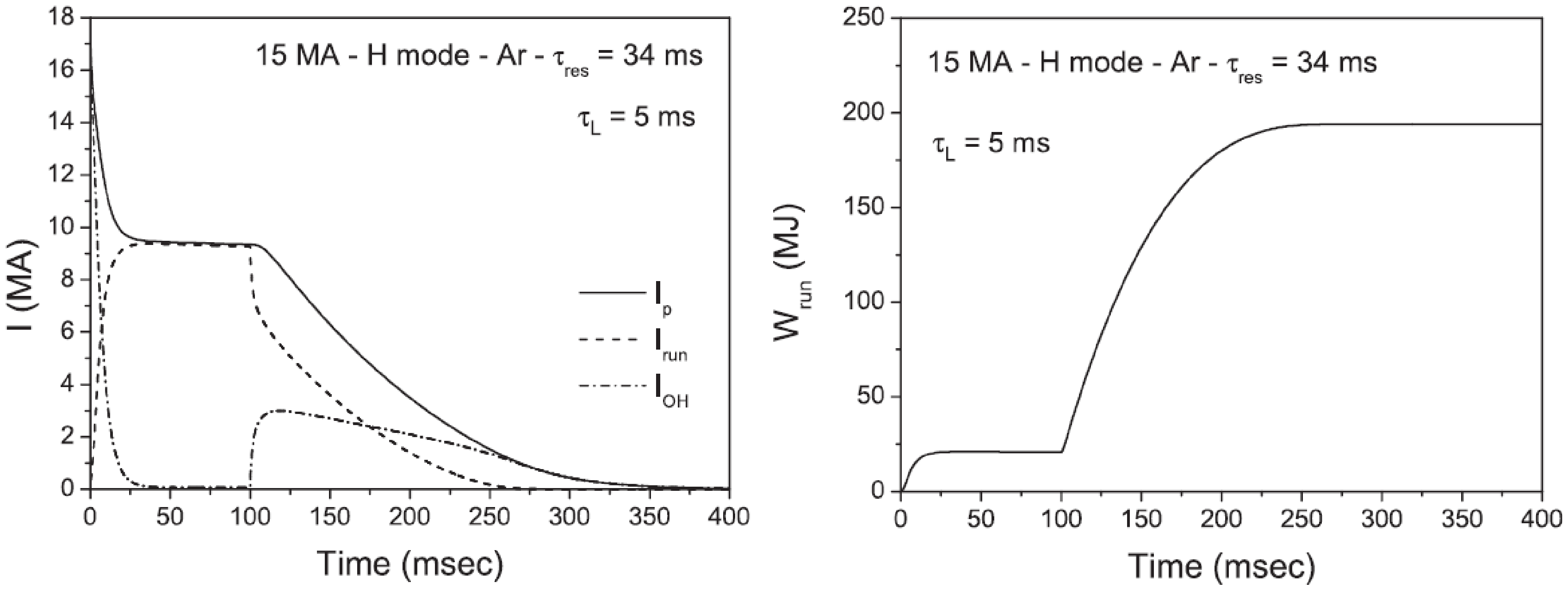


Figure 29. For a 15 MA H-mode disruption with Ar injection and $\tau_{res} = 34$ ms: left: time evolution of the total plasma current, runaway current and ohmic current during the current quench and termination phases of the disruption. The termination phase starts at 100 ms and $\tau_L = 5$ ms. Right: time evolution of runaway kinetic energy gain.

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Grad-Shafranov equation

- Let's take a look at the ordinary Grad-Shafranov equation before we venture forth to construct the RE equilibrium.
- There are essentially two ingredients in the GS equation:
  - The **symmetry** of the system, toroidal symmetry in tokamaks.
  - The **force balance** equation:

$$\boldsymbol{J} \times \boldsymbol{B} = \boldsymbol{\nabla} p, \qquad \boldsymbol{B} = \boldsymbol{\nabla}\Psi \times \boldsymbol{\nabla}\phi + \bar{F}\boldsymbol{\nabla}\phi$$

$$\mu_0 \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B} = -\Delta^* \Psi \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\bar{F} \times \boldsymbol{\nabla}\phi$$

- The operator $\Delta^* \equiv R \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{\partial^2}{\partial Z^2}$
- Rewrite the force balance equation as

$$-\Delta^* \Psi \boldsymbol{\nabla}\Psi \frac{1}{R^2} - \frac{\boldsymbol{\nabla}\bar{F}^2}{2R^2} = \mu_0 \boldsymbol{\nabla} p$$

- Realizing that both p and $\bar{F}$ are functions of Ψ alone, we arrive at the **Grad-Shafranov equation**:

$$\Delta^* \Psi = -\frac{1}{2}\frac{d\bar{F}^2}{d\Psi} - \mu_0 R^2 \frac{dp}{d\Psi}$$

Grad-Shafranov equation

- The solution of the Grad-Shafranov equation enjoys some good properties:

$$\boldsymbol{B}\cdot\nabla\Psi=0,\qquad \boldsymbol{J}\cdot\nabla\Psi=0,\qquad \nabla p\times\nabla\Psi=0$$

- The contours of the magnetic flux Ψ act as the **characteristic surface** of the plasma.
- High particle parallel momentum would result in deviation between their **trajectory line** and the magnetic **field line**. For REs, the current does not flow along the flux surface anymore!
- **The ordinary GS equation is not valid for RE equilibria!**
- **More accurate equilibrium theory is needed.**

Status of RE equilibria theories

- Last year, we, Vinodh & Matthias approached this problem independently from slightly different ways.
- One could either essentially add the **centrifugal force** of the runaway electrons to the old force balance equation, then solve for the magnetic flux with the additional terms.

$$p_\parallel \boldsymbol{\kappa} = \boldsymbol{J} \times \boldsymbol{B} - \nabla_\perp p_{\text{th}}, \qquad \boxed{\Delta^* \psi = -\mu_0 R^2 p'_{\text{th}} - R B_\phi F' + \mu_0 R (e n_r v_\parallel).}$$

- Or solve for the **drift surface** of the REs directly

$$\boldsymbol{A}^* \equiv \boldsymbol{A} + \frac{p_\parallel}{q} b, \quad \boldsymbol{B}^* \equiv \nabla \times \boldsymbol{A}^*, \quad \boldsymbol{B}^* = \nabla \Psi^* \times \nabla \phi + \bar{F}^* \nabla \phi. \quad \boldsymbol{J} \times \boldsymbol{B}^* = 0.$$

$$\Delta^* \Psi^* = -\mu_0 n_{\text{RE}} q R \frac{p_\parallel}{\gamma m} - \frac{p_\parallel}{qR}.$$

L. Yuan et al., Chin. Phys. B **32** 075208 (2023); V. Bandaru et al., Phys. Plasmas **30**, 092508 (2023)

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The GS-like equilibrium

- **The toroidal symmetry still hold** for the 2D equilibrium sustained by the RE current……
- But we need **a new force balance equation** for the runaway electrons.
- Essentially this involves adding **the centrifugal force** term into the force balance equation.
- Formally, one could obtain the equation of motion of individual RE from the Lagrangian mechanics, then perform an integral over the momentum space to get a fluid representation of the "RE fluids" and the corresponding force balance equation.

The guiding center EoM

- From the guiding center Lagrangian $L(\boldsymbol{X}, p_{\|}, \mu, \theta)$ we obtain the Equation of Motion for individual RE guiding centre:

$$\dot{\mathbf{X}} = \frac{p_{\|}}{\gamma m}\frac{\mathbf{B}^*}{B_{\|}^*} + \frac{\mathbf{b}}{qB_{\|}^*} \times \left(\nabla\Phi^* + \frac{\partial \mathbf{A}^*}{\partial t}\right),$$

$$\dot{p_{\|}} = -\frac{\mathbf{B}^*}{B_{\|}^*} \cdot \left(\nabla\Phi^* + \frac{\partial \mathbf{A}^*}{\partial t}\right),$$

$$\dot{\mu} = 0,$$

$$\dot{\theta} = \frac{qB}{\gamma m}.$$

$$\mathbf{J} = n_{RE} q \mathbf{u}.$$

$$\mathbf{A}^* \equiv \mathbf{A} + \frac{p_{\|}}{q}\mathbf{b},$$

$$\mathbf{B}^* \equiv \nabla \times \mathbf{A}^*,$$

$$B_{\|}^* \equiv \mathbf{B}^* \cdot \mathbf{b},$$

$$\nabla\Phi^* \equiv \frac{\mu \nabla B}{\gamma} + q\nabla\Phi.$$

- A few preliminary simplification could be carried out.

Cary, John R, Brizard, and Alain J 2009 Rev. Mod. Phys **81** 693-738;
Liu C, Qin H, Eero Hirvijoki, Wang Y L, and Liu J 2018 Nucl. Fusion **58** 106018

Simplification of the RE EoM

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- We assume the following ordering: $\frac{p_\parallel}{qBR} \sim \varepsilon^2$, the higher order curvature terms and their modification to the magnetic moment μ are neglected.
- We assume steady state, strongly passing electrons, hence we have $\mu \to 0$, $\nabla\Phi^* \cong 0$, and $\frac{\partial}{\partial t} \to 0$.
- δ function distribution in the momentum space, $p_\parallel = const$.

$$\dot{\mathbf{X}} = \frac{p_\parallel}{\gamma m}\frac{\mathbf{B}^*}{B_\parallel^*} + \frac{\mathbf{b}}{qB_\parallel^*} \times \left(\nabla\Phi^* + \frac{\partial \mathbf{A}^*}{\partial t}\right),$$

$$\dot{p_\parallel} = -\frac{\mathbf{B}^*}{B_\parallel^*} \cdot \left(\nabla\Phi^* + \frac{\partial \mathbf{A}^*}{\partial t}\right),$$

→

$$\dot{\mathbf{X}} = \frac{p_\parallel}{\gamma m}\frac{\mathbf{B}^*}{B_\parallel^*},$$

$$\dot{p_\parallel} = 0.$$

New force balance equation

- The above simplification yields:

$$\mathbf{J} = n_{RE} q \frac{p_\parallel}{\gamma m} \frac{\mathbf{B}^*}{B_\parallel^*}.$$

- Apparently, such current satisfy the following force balance equation:

$$\mathbf{J} \times \mathbf{B}^* = 0.$$

- While we could define the new characteristic surface of the REs as using the reduce representation:

$$\mathbf{B} = \nabla\Psi \times \nabla\phi + \bar{F}\nabla\phi, \qquad \mathbf{B}^* \cdot \nabla\Psi^* = 0,$$

$$\mathbf{B}^* = \nabla\Psi^* \times \nabla\phi + \bar{F}^*\nabla\phi. \qquad \mathbf{J} \cdot \nabla\Psi^* = 0.$$

GS-like equilibrium

- Using the new force balance equation, we follow the same method of deriving the GS equation, and obtain:

$$\mu_0 \boldsymbol{J} = \nabla \times \boldsymbol{B} = -\Delta^* \Psi \nabla \phi + \nabla \bar{F} \times \nabla \phi$$

+

$$\boldsymbol{J} \times \boldsymbol{B}^* = 0, \qquad \boldsymbol{B}^* = \nabla \Psi^* \times \nabla \phi + \bar{F}^* \nabla \phi$$

↓

$$-\Delta^* \Psi \nabla \Psi^* \frac{1}{R^2} - \frac{\bar{F}^* \nabla \bar{F}}{R^2} = 0$$

- Realizing $\bar{F}$ should be a function of Ψ^* alone, we finally have $\Delta^* \Psi = -\frac{d\bar{F}}{d\Psi^*} \bar{F}^*$. However, the RHS is not a function of Ψ alone…… How to solve this?

GS-like equilibrium

- We realize the relationship between the flux surface and the RE drift surface.

$$\nabla\Psi = \nabla\Psi^* - \frac{p_\parallel}{q}\nabla R, \qquad \Delta^*\Psi = \Delta^*\Psi^* + \frac{p_\parallel}{qR}$$

- This is consistent with the definition of $\boldsymbol{A}^* \equiv \boldsymbol{A} + \frac{p_\parallel}{q}\boldsymbol{b}$
- The previous equilibrium equation can now be re-written:

$$\Delta^*\Psi = -\frac{d\bar{F}}{d\Psi^*}\bar{F}^*. \quad + \quad \Psi = \Psi^* - \frac{p_\parallel}{q}R. \quad \Rightarrow \quad \Delta^*\Psi^* = -\mu_0 n_{RE} qR\frac{p_\parallel}{\gamma m} - \frac{p_\parallel}{qR}.$$

- The differential equation could be solved numerically given the boundary condition and known RHS profiles.

Published in *L. Yuan and D. Hu, 2023 Chin. Phys. B* ***32****, 075208*;
Independently obtained in *V. Bandaru and M. Hoelzl, Phys. Plasmas* ***30****, 092508 (2023)* as a special case in their multi-fluid equilibria.

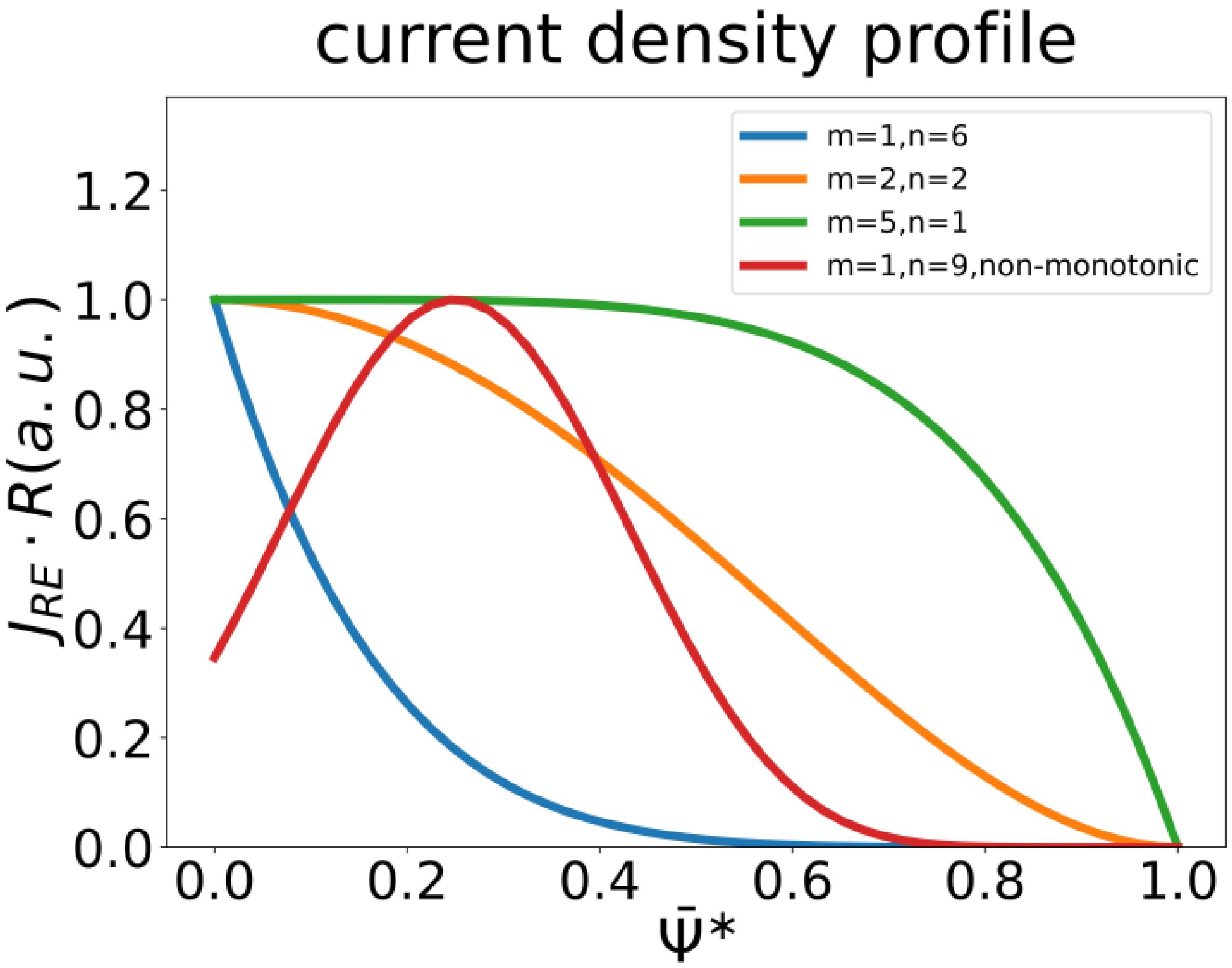
Numerical solution

- We consider two sets of simple 2D geometry: the MAST-like and the ITER-like geometries.
- Simple rectangular boundary is assumed, and the ideal boundary condition is used (with prescribed, frozen-in vertical field or poloidal field).
- Code based on **Free-GS**.
- The total RE current is set to 200kA or 1MA. The current profile have the following form:

$$J_{\|} R = n_{RE} q \frac{p_{\|}}{\gamma m} R,$$

$$n_{RE} R = n_{RE}(0) R_0 \left(1 - \bar{\psi}^{*m}\right)^n. \quad \text{or} \quad n_{RE} R = n_{RE}(0) R_0 \left[1 - \bar{\psi}^{*2} + \left(\bar{\psi}^{*} - \bar{\psi}^{*2}\right)^m\right]^n.$$

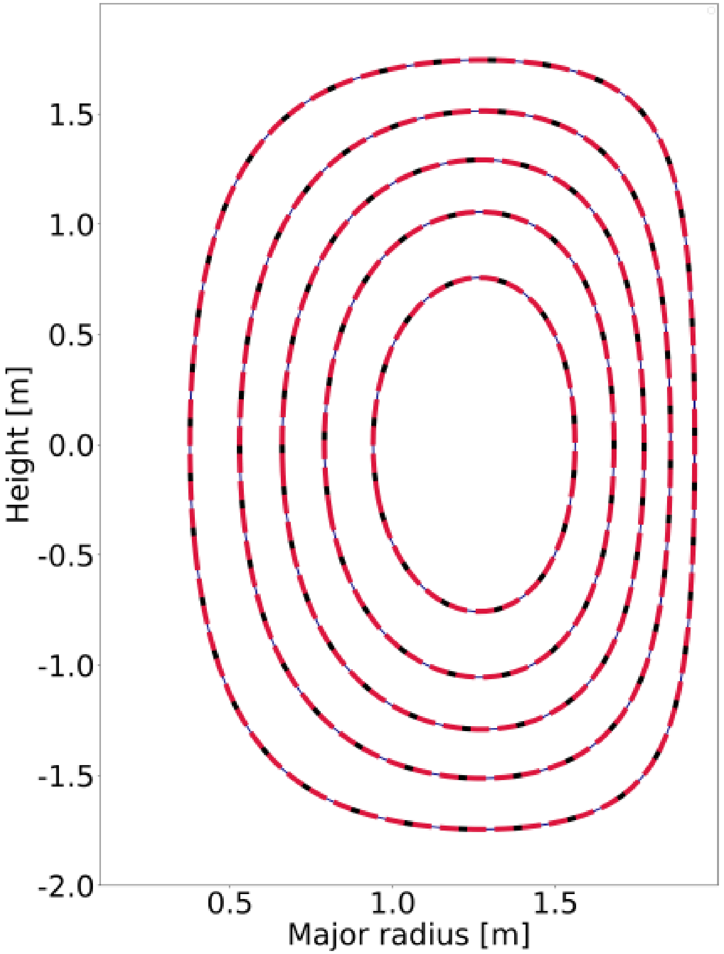
RE current profiles

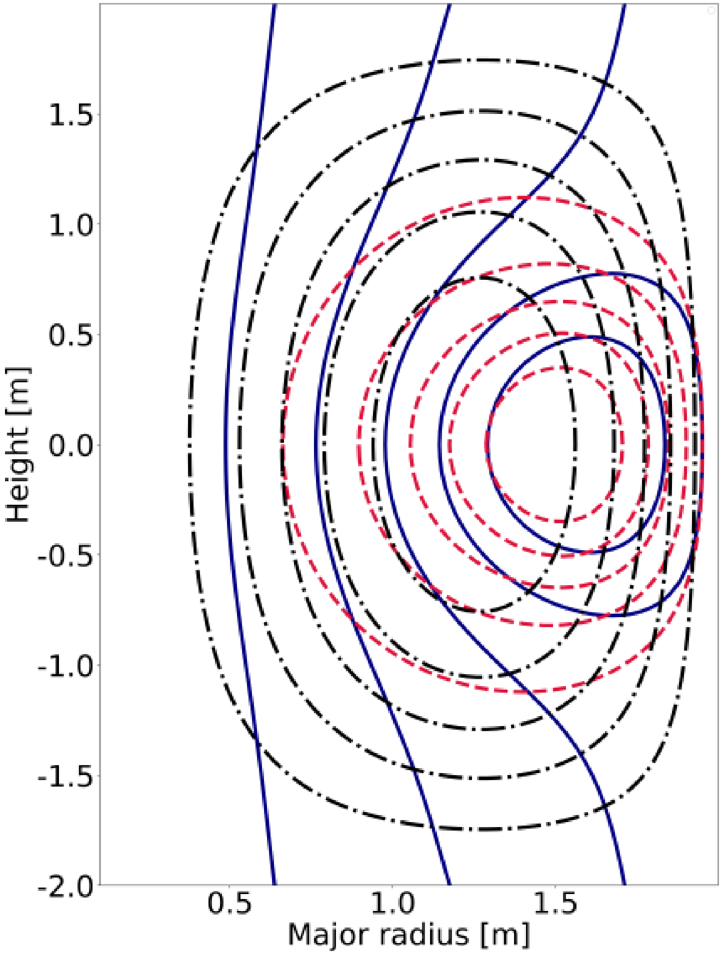


Benchmark case, MAST-like

$I_p = 200kA$
$B_Z = 0$

$p_{\parallel} = 0$

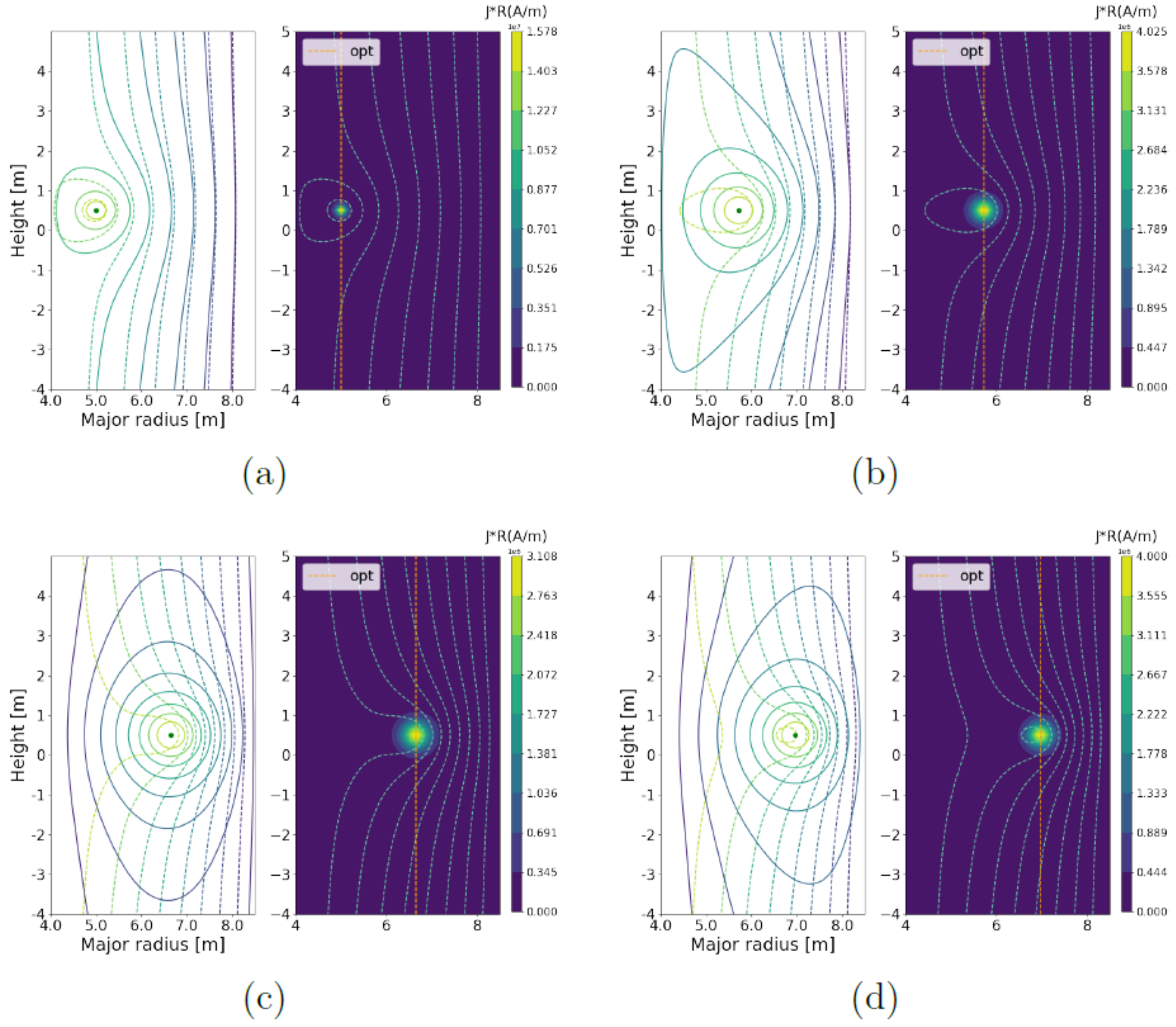




$p_{\parallel} = 20mc$

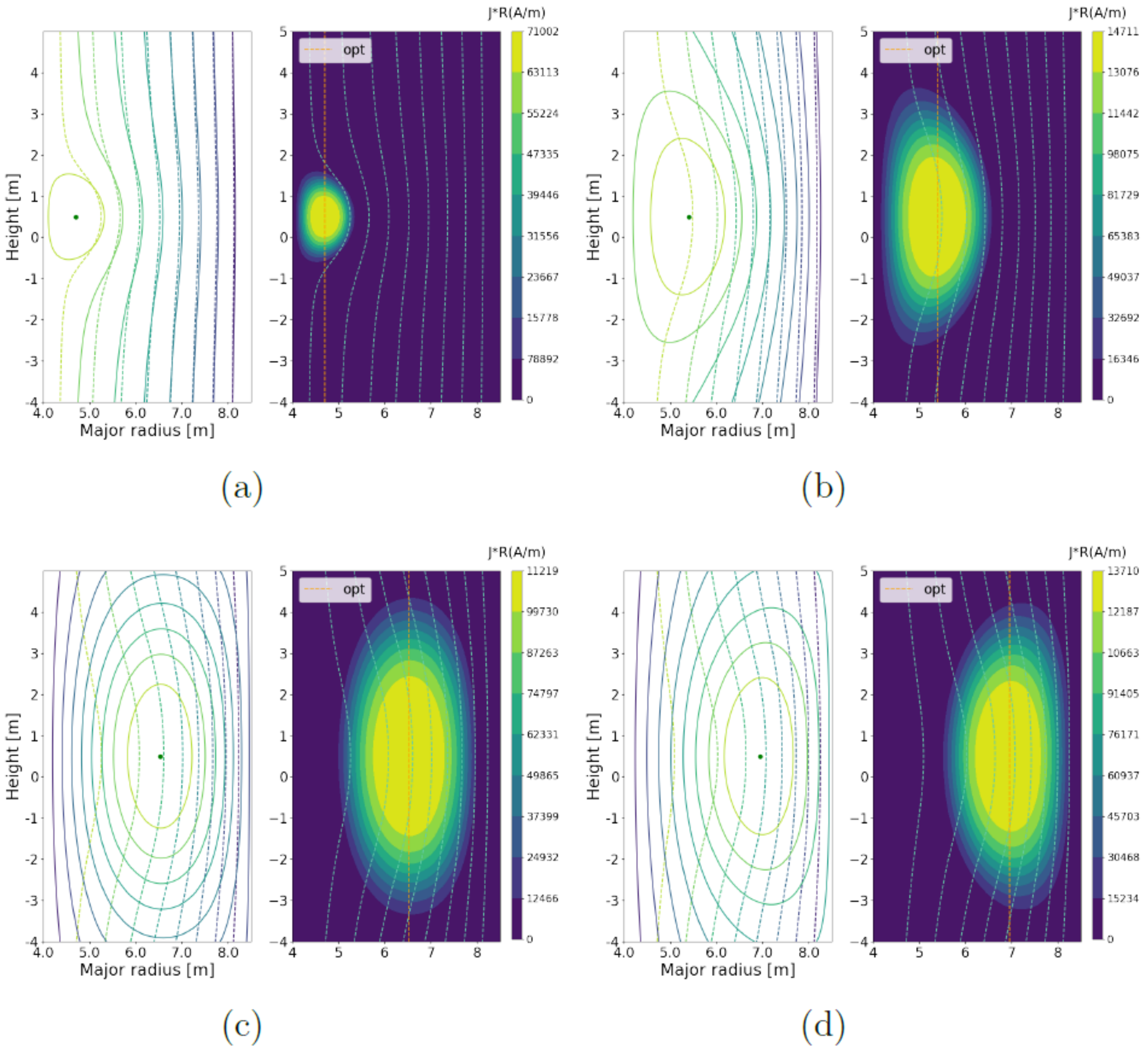
Examples of numerical solutions

$$I_p = 200kA$$
$$B_Z = -0.045T$$
$$m = 1$$
$$n = 6$$
$$\gamma = 56, 116, 176, 200$$



Examples of numerical solutions

$$I_p = 200kA$$
$$B_Z = -0.045T$$
$$m = 2$$
$$n = 2$$
$$\gamma = 56, 116, 176, 200$$



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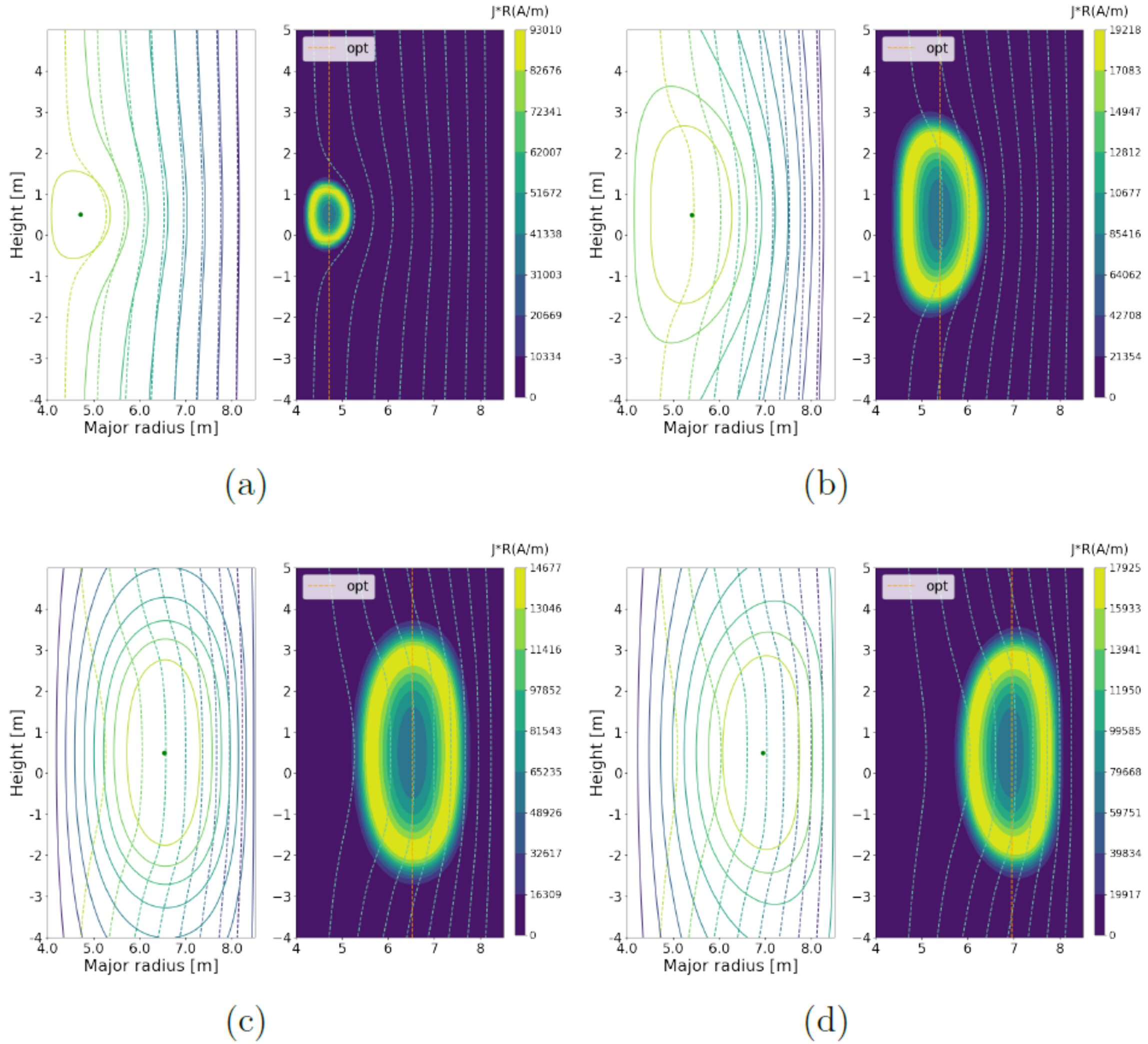
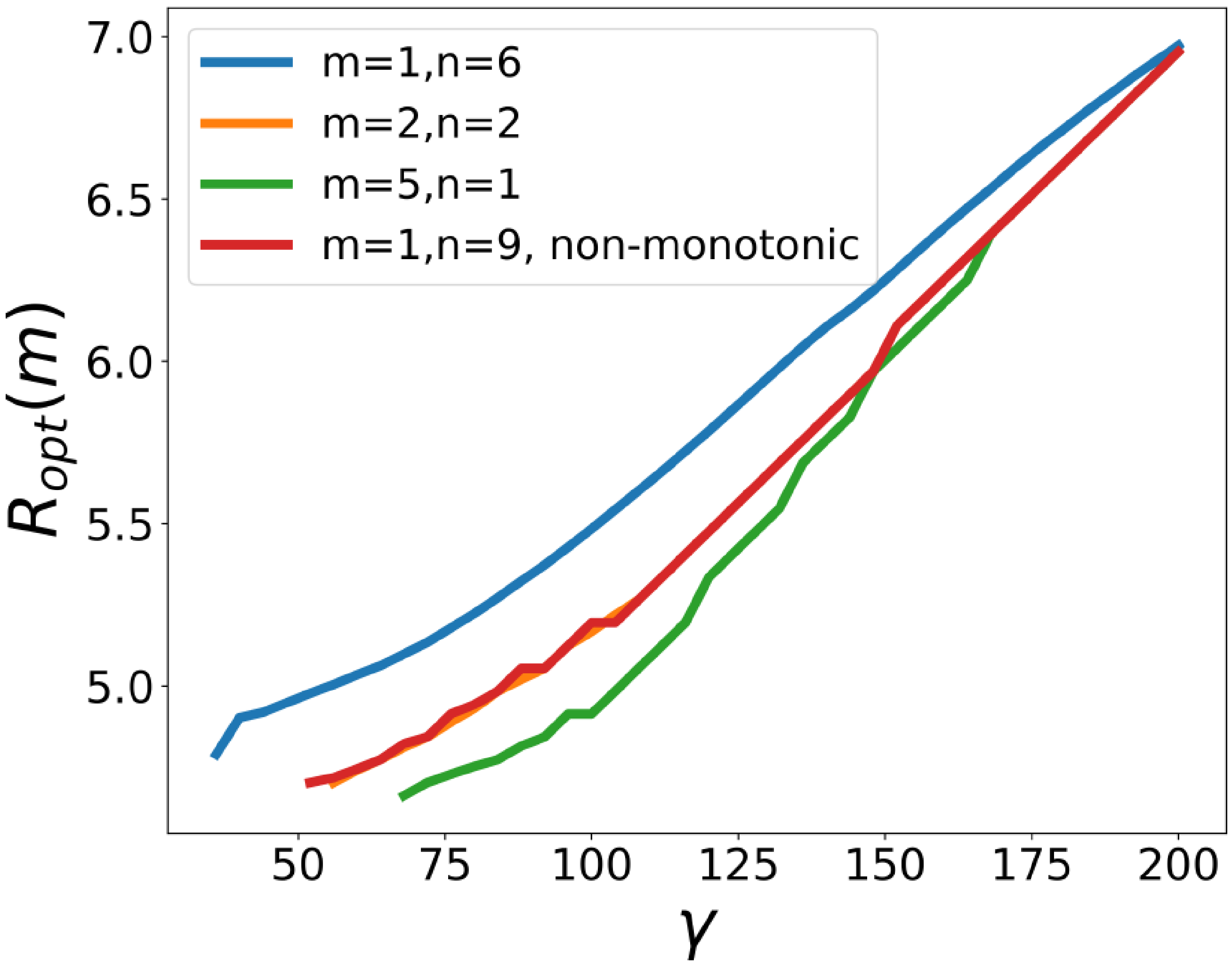
Examples of numerical solutions

$I_p = 200kA$
$B_Z = -0.045T$
Non-monotonic
$\gamma = 56, 116, 176, 200$



Current center displacement

$I_p = 200kA$
$B_Z = -0.045T$

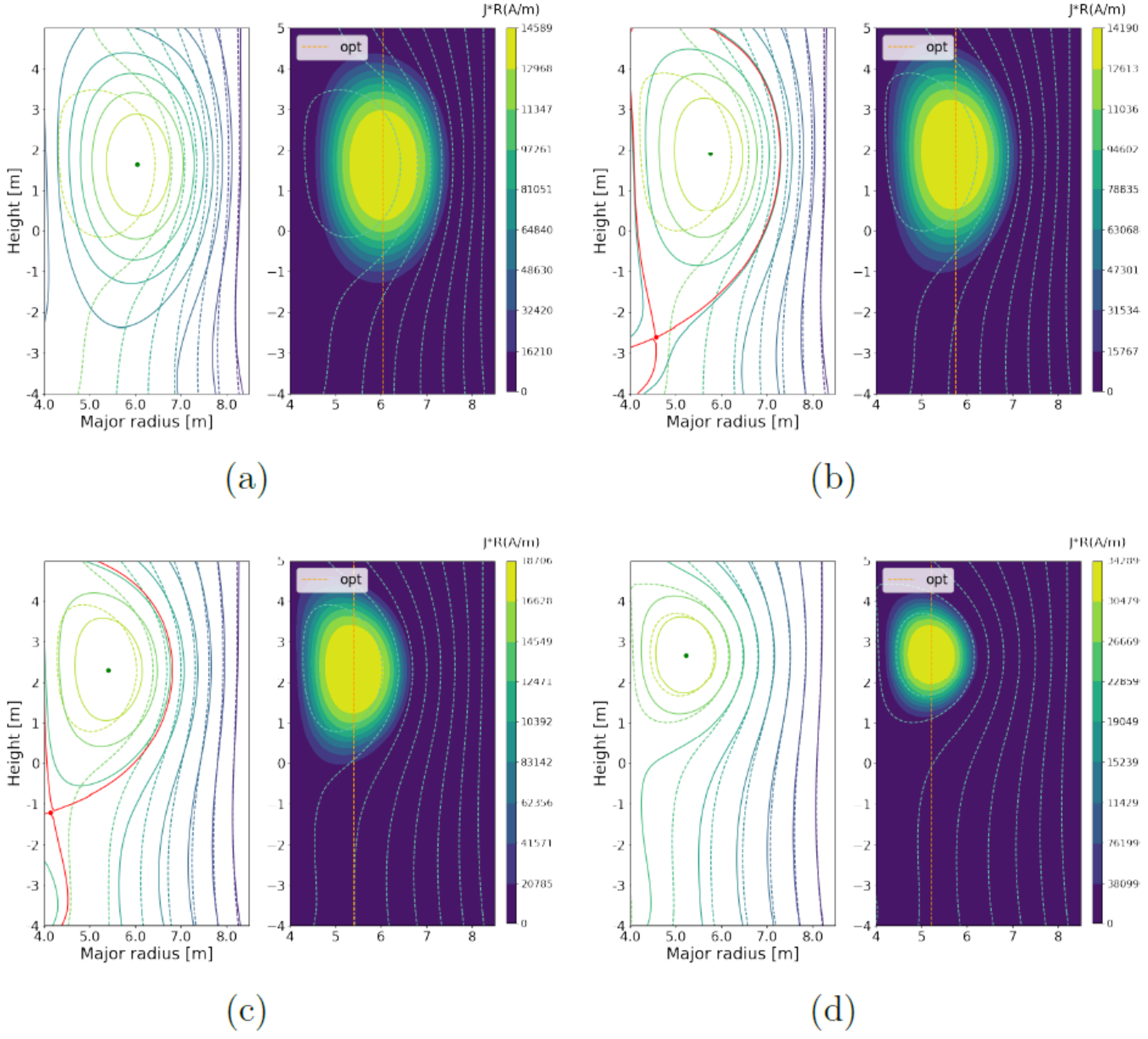


Possible vertical displacement

Simple up-down asymmetric external poloidal coil set up:

$$x_u = 5m,\ y_u = 7.75m$$
$$x_d = 6m,\ y_d = -7.75m$$
$$I_u = 85.94kA$$
$$I_d = 77.35kA$$

$$I_p = 200kA$$
$$B_{z0} = -0.018T$$



Potential experimental evidence?

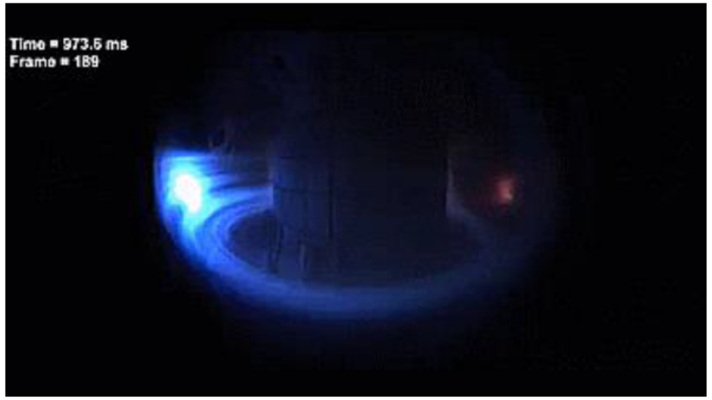
COMPASS

What diagnostics are available?
Radiation diagnostic for RE density diagnostics?
Momentum space distribution?

IPP

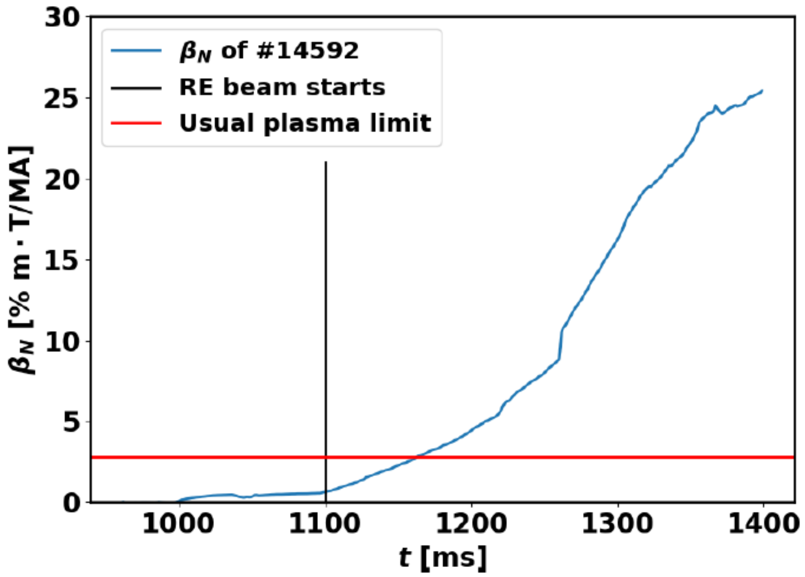
THERE IS SOMETHING WRONG WITH THESE BEAMS… OR EFIT… AND POSITION CONTROL…

RE beams generated during CQ quickly expand/drift outwards in the plateau phase (e.g. #12202)



Video courtesy of A. Havranek, V. Weinzettl, J. Cavalier

Beta normalized reaches values 10x larger that what should be achievable in the tokamak



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O. Ficker et al., REM 2023.

How to go forward?

- The natural step forward is to admit variation of the parallel momentum and to consider its distribution.

$$\boldsymbol{A}^* = \boldsymbol{A} + \frac{p_\parallel}{q}\hat{\boldsymbol{b}} \rightarrow \boldsymbol{A}^\dagger = \boldsymbol{A} + \frac{\langle p_\parallel \rangle(\boldsymbol{x})}{q}\hat{\boldsymbol{b}}$$

- **Problem: the average of B^* is not divergence free!**
- Define the effective $\boldsymbol{B}$ field from $\boldsymbol{A}^*$ instead

$$\boldsymbol{B}^\dagger \equiv \nabla \times \boldsymbol{A}^\dagger = \boldsymbol{B} + \frac{\langle p_\parallel \rangle}{q}\nabla \times \hat{\boldsymbol{b}} + \frac{\nabla\langle p_\parallel \rangle \times \hat{\boldsymbol{b}}}{q} = \langle \boldsymbol{B}^* \rangle + \frac{\nabla\langle p_\parallel \rangle \times \hat{\boldsymbol{b}}}{q}$$

$$\boldsymbol{J} = \int f q \dot{\boldsymbol{X}} \mathrm{d}p_\parallel \approx qc \int \frac{\boldsymbol{B}^*}{B_\parallel^*} f \mathrm{d}p_\parallel \approx n_{RE} qc \frac{\langle \boldsymbol{B}^* \rangle}{B}$$

$$\boldsymbol{J} \times \langle \boldsymbol{B}^* \rangle = 0 \rightarrow \boxed{\boldsymbol{J} \times \boldsymbol{B}^\dagger \approx n_{RE}(\Psi^\dagger) c \nabla \langle p_\parallel \rangle (\Psi^\dagger)}$$

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Conclusion & Discussion

- We obtained the **new force balance equation** for the strongly passing REs from their EoM, which in turn is obtained from their Lagrangian.
- With the new force balance equation and the toroidal symmetry, a **new Grad-Shafranov-like equilibrium equation** is obtained for the RE current dominant equilibria.
- Numerical solution of this new equilibria shows **horizontal current center displacement** caused by momentum change of the current carrier, even when the current itself remain the same. Implication to CQ RE current position control.
- With up-down asymmetry in the external poloidal field, **vertical displacement** accompanies the horizontal one, implication to CQ VDE.
- More physics to be added, finite momentum space distribution width, configuration space distribution of the RE parallel momentum, pitch angle, higher $p_{\|}/qBR$, realistic coils & wall…

Backup slides

GS-like equilibrium

- There are two ways to proceed:
  - Rewrite the RHS as a function of Ψ and some configuration space coordinates such as R etc.. (*Multi-fluid equilibria*……)
  - Rewrite the LHS as a function of Ψ^*, while write the RHS as a function of Ψ^* and some configuration space coordinates.
- As Ψ^* corresponds to the canonical angular momentum of the REs, it might be easier to pursue the second option.
- To obtain the relationship between $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}^*$, as well as $\bar{\boldsymbol{F}}$ and $\bar{\boldsymbol{F}}^*$, we write:

$$\boldsymbol{B}^* = \nabla \times \boldsymbol{A}^* = \nabla \times \left(\boldsymbol{A} + \frac{p_\parallel}{q}\boldsymbol{b}\right) = \nabla(\Psi + \zeta) \times \nabla\phi + (\bar{F} + \bar{G})\nabla\phi$$

? ?

- Apparently, $\bar{G} = R^2\left(\nabla \times \frac{p_\parallel}{q}\boldsymbol{b}\right) \cdot \nabla\phi$, $\nabla\zeta = -R^2\left(\nabla \times \frac{p_\parallel}{q}\boldsymbol{b}\right) \times \nabla\phi$.

GS-like equilibrium

- In this first attempt, we assume $p_{\parallel}$ is also a constant across the configuration space. Further work aiming at relaxing this assumption is well underway. We have:

$$\bar{G} = R^2\left(\nabla \times \frac{p_{\parallel}}{q}\boldsymbol{b}\right)\cdot\nabla\phi = R^2\frac{p_{\parallel}}{q}\left(\frac{\nabla\times\boldsymbol{B}}{B} - \frac{\nabla B\times\boldsymbol{B}}{B^2}\right)\cdot\nabla\phi$$

$$= R^2\frac{p_{\parallel}}{q}\left(-\frac{\Delta^*\Psi}{BR^2} - \frac{\nabla B\times\boldsymbol{B}}{B^2}\cdot\nabla\phi\right)$$

- To the leading order, $\nabla B \cong -\frac{1}{R}B\nabla R$
- We could write down the following in (R,Z) coordinates

$$\bar{G} = \frac{p_{\parallel}}{qB}\left[-R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right) - \frac{1}{R}\frac{\partial\Psi}{\partial R} - \frac{\partial^2\Psi}{\partial Z^2}\right] = -\frac{p_{\parallel}}{qB}\left[R\frac{\partial B_Z}{\partial R} + B_Z + \frac{\partial^2\Psi}{\partial Z^2}\right]$$

- Realizing $R\frac{\partial B_Z}{\partial R}/B_Z \sim \varepsilon^{-1}$, hence we could approximately write $\bar{G} \cong -\frac{p_{\parallel}}{qB}\Delta^*\Psi.$

GS-like equilibrium

- Similarly, we have

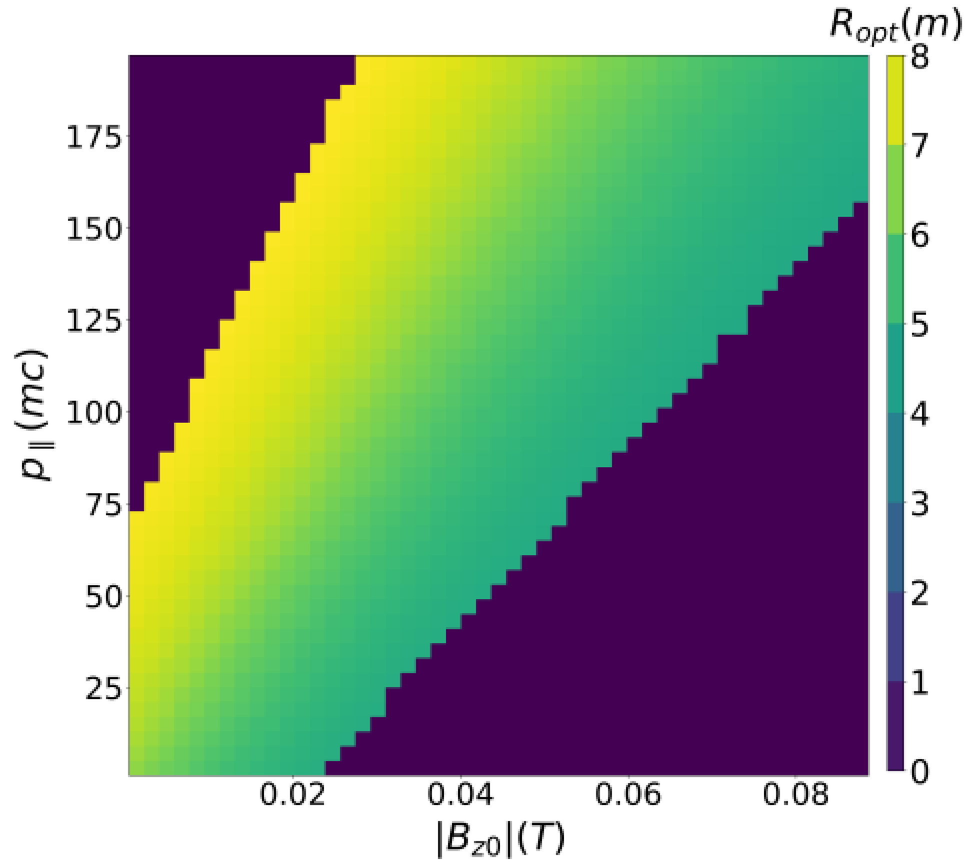
$$\nabla\zeta = -R^2\left(\nabla\times\frac{p_\parallel}{q}\boldsymbol{b}\right)\times\nabla\phi = R^2\frac{p_\parallel}{q}\left[\frac{\nabla\bar{F}}{BR^2}+\frac{\bar{F}\nabla R}{BR^3}\right]$$
$$=\frac{p_\parallel}{qB}\left[\nabla\bar{F}+\frac{\bar{F}}{R}\nabla R\right]$$

- Realizing $|R\nabla\bar{F}|\ll\bar{F}$ and $\frac{\bar{F}}{B}=R+\mathcal{O}(\varepsilon^2)$, we have

$$\nabla\zeta\cong\frac{p_\parallel}{q}\nabla R$$

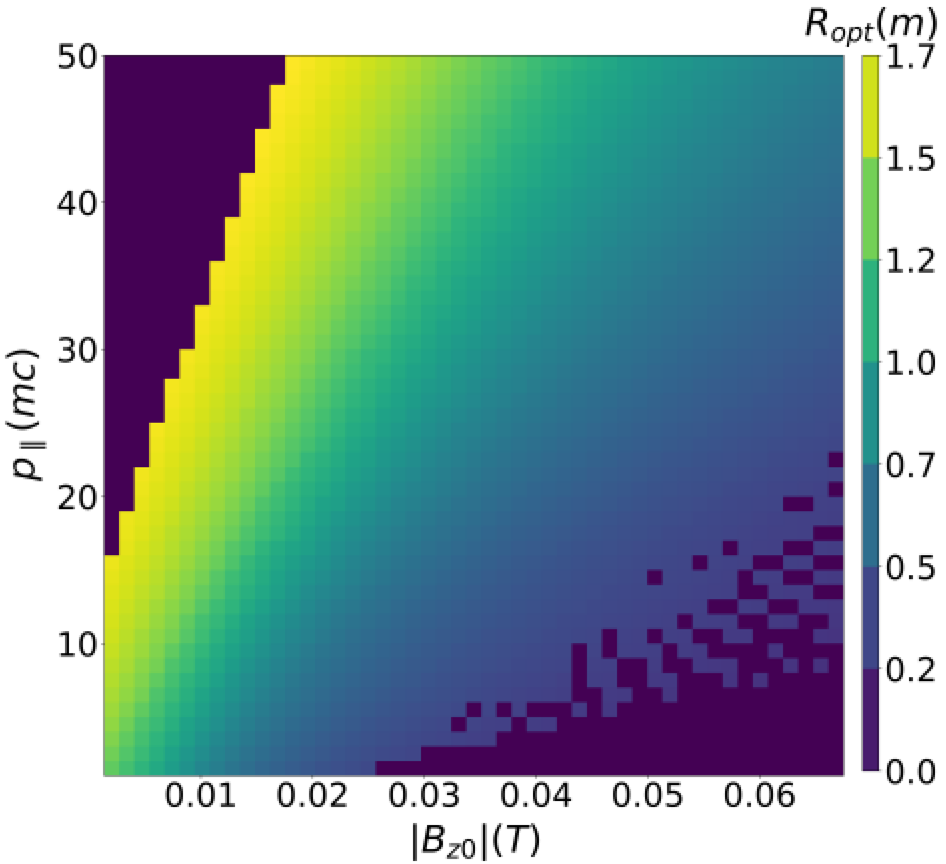
Geometric dependency

$I_p = 200kA$



ITER-like

(a)



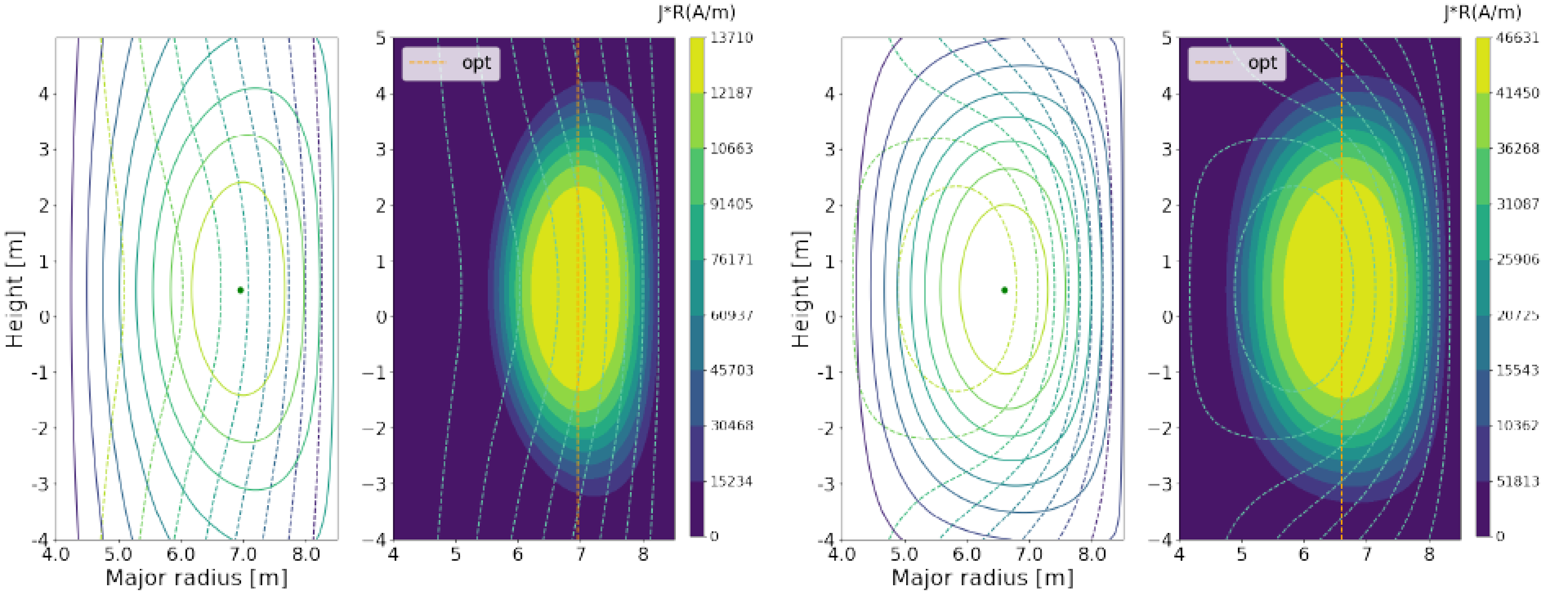
(b)

MAST-like

Magnetic field dependency

$I_p = 200kA$

$I_p = 1MA$



(a) (b)