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Finite orbit width stabilizing effect on toroidal Alfvén eigenmodes excited by passing and trapped energetic ions

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Abstract. A general expression for the growth rate of the toroidal Alfvén eigenmode (TAE) instability is derived by employing a Hamiltonian action-angle approach. It takes into account the finite orbit width effects, the finite Larmor radius (FLR) effects of energetic particles and the effect of mode localization, as well as the possible mode excitation by both passing and trapped energetic ions. In particular, the stabilizing effect of the large particle orbit widths on the TAE modes excited by fast passing and trapped ions is investigated. In the limiting cases when $\Delta_m^{(i)} \ll \Delta_b^t \ll \Delta_m^{(o)}$ and $\Delta_b^t \gg \Delta_m^{(o)}$, the instability drive is reduced by factors of $\Delta_m^{(i)}/\Delta_b$ and $\Delta_m^{(i)}(\Delta_m^{(o)})^2/\Delta_b^3$, respectively, as compared with the prediction of the narrow orbit theory, when $\Delta_b^t \ll \Delta_m^{(i)}$, where Δ_b is the particle orbit width and $\Delta_m^{(i)}$ ($\Delta_m^{(o)}$) is the inner (outer) width of the mode structure. For the TFTR parameters, the stabilization effect is much stronger in the case of TAE excitation by fusion-produced alpha particles than that in connection with neutral beam injection (NBI)-generated energetic ions.

1. Introduction

Large tokamak experiments such as JET and TFTR have recently started operation using fuel mixtures with deuterium and tritium in order to test the physics of the fusion-produced energetic alpha particles. The successful confinement of alphas during slowing time scales is a crucial requirement for the operation conditions in ignited regimes of future tokamak reactors such as ITER. In addition to fusion-produced alpha particles, energetic ion populations can be present from neutral beam injection (NBI) for heating or current drive, as well as from ion cyclotron resonance heating (ICRH).

The determination of plasma instabilities arising due to the presence of these energetic particle populations in reactor relevant regimes has become one of the major problems to be solved for future toroidal magnetic fusion devices. In the case of alphas and other fast ion populations having their initial birth velocities higher than the Alfvén speed, the expansion energy associated with the density gradient of fast ions can drive shear Alfvén modes unstable via wave–particle resonant interaction, [1, 2]. These shear Alfvén waves normally experience significant Landau damping due to their short radial wavelengths.

However, an exception occurs with the toroidicity-induced shear Alfvén eigenmodes (TAE modes), which are discrete waves located in the toroidicity-induced spectral gaps between the shear Alfvén continua, [3–5]. The TAE modes can be strongly excited by energetic particles if the mode frequency ω_A is lower than the diamagnetic frequency of the fast ions, $\omega_{\star\alpha}$, [6–18]. The TAE instability is, at present, considered to be one of the most

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serious thermonuclear instabilities because of its low hot-beta threshold and global mode structure, but also because it can cause rapid anomalous loss of energetic alpha particles and thus result in degradation of the plasma performance and damage on the first wall [19].

Several theoretical models have predicted the presence of TAE instability in ignited tokamaks such as ITER [9, 20]. The quasilinear saturation of a single mode of this instability has also been studied [21]. Recent simulating experiments on TFTR [22–24], DIII-D [25, 26], JET [27, 28] and JT-60U [29], with energetic ions from NBI and/or those created by ICRH have demonstrated excitation of the TAE modes. In particular, the excitation of the TAE modes was coupled to significant fast particle losses in the DIII-D experiment.

However, the experimentally observed instability thresholds have generally been higher than theoretical predictions. This has motivated interest in the accurate evaluation of various damping mechanisms for the TAE instability, such as electron and ion Landau damping [8–11], Alfvén continuum damping [30–32], and collisional damping on trapped electrons [33]. On the other hand, most of the existing analytical theories have been based on the restriction that the radial excursion of the energetic particle orbits from the flux surfaces, Δ_{b} (the so-called particle orbit width) is small compared with the TAE localization scale length, Δ_m. However, this assumption is generally violated and it was shown analytically [13, 17] and numerically [14] that the energetic particle drive for the TAE instability excited by passing ions is significantly reduced when $\Delta_m/\Delta_{\mathrm{b}} \ll 1$. Furthermore, numerical simulations have shown that a similar stabilization effect takes place in the case of TAE excitation by trapped ions [14, 16]. Additional stabilization is provided by the FLR effect as shown in [18].

The purpose of the present paper is to derive a generalized expression for the linear instability growth rate of the TAE modes which takes into account the large particle orbit width effects and the effect of mode localization, as well as the excitation by both passing and trapped energetic ions. In order to clearly demonstrate the role of the large particle orbit width effects for the TAE instability drive, a comparison will be made between the cases when $\Delta_{\mathrm{b}} \ll \Delta_m$ and $\Delta_{\mathrm{b}} \gg \Delta_m$. In addition, we will show that application of the developed theory to the parameters of NBI heating experiment on TFTR predicts TAE growth rates being in a good agreement with the results from numerical simulations of [14]. In the case of TAE excitation by fusion-produced alpha particles the stabilization effect due to large orbit width is found to be much stronger than that in connection with NBI-generated fast ions.

The outline of the paper is as follows. In section 2, the basic eigenvalue equation system describing the linear evolution of the TAE modes in a plasma with energetic ions is derived following the technique outlined in [2]. The kinetic response of the energetic ions is determined in section 3 and the generalized instability growth rate for the TAE modes is calculated perturbatively in section 4. Here, we also discuss the particle orbit width effect on the instability growth rate. In section 5 we apply the obtained results to the cases of NBI-generated fast ions and an isotropic distribution of fusion-produced alpha particles for TFTR parameters. Finally, section 6 is devoted to conclusions.

2. Basic equations for TAE modes

The derivation of the eigenvalue equations describing linear evolution of the TAE modes in the presence of energetic particles follows here the technique outlined by Rosenbluth and Rutherford [2] (see also [6, 30, 32]). We shall consider a low-beta plasma in a large aspect-ratio tokamak with circular cross section. Assuming the linearized field variables $\boldsymbol{X}$ to be in the form $\boldsymbol{X} = \boldsymbol{X}_0 + \tilde{\boldsymbol{X}}$, where $\boldsymbol{X}_0$ is the equilibrium part and $\tilde{\boldsymbol{X}}$ is a small perturbation, we begin the derivation of the shear Alfvén eigenmode equations from the condition for charge neutrality implying that

$$\nabla\cdot\nabla\cdot\tilde{\boldsymbol{j}}_\perp+\nabla\cdot\left(\frac{\boldsymbol{B}_0}{B_0}\tilde{j}_\parallel\right)+\nabla\cdot\tilde{\boldsymbol{j}}^k=0 \tag{1}$$

where $\boldsymbol{B}_0$ is the equilibrium magnetic field vector, $\nabla\cdot\tilde{\boldsymbol{j}}_\perp$ and $\tilde{\boldsymbol{j}}_\perp$ are the perpendicular and parallel components, respectively, of the perturbed current density due to the magnetohydrodynamic plasma motion, and $\tilde{\boldsymbol{j}}^k$ is the perturbed current density induced by kinetic wave–particle interaction.

In order to obtain $\nabla\cdot\tilde{\boldsymbol{j}}_\perp$, we use the linearized momentum balance equation

$$-\mathrm{i}\omega c\rho_0\tilde{\boldsymbol{u}}=\boldsymbol{j}_0\times\tilde{\boldsymbol{B}}+\tilde{\boldsymbol{j}}_\perp\times\boldsymbol{B}_0 \tag{2}$$

where $\tilde{\boldsymbol{u}}$ is the fluid plasma velocity, the perturbed quantities are assumed to have a time variation as $\mathrm{e}^{-\mathrm{i}\omega t}$, the perturbation of the mass density, ρ, has been ignored by employing charge neutrality and the plasma pressure perturbation has been neglected by adopting the low-beta approximation. From the linearized ideal Ohm's law we have

$$\tilde{\boldsymbol{u}}=c\frac{\tilde{\boldsymbol{E}}\times\boldsymbol{B}_0}{B_0^2} \tag{3}$$

and the electric field, $\boldsymbol{E}$, may be represented in terms of the scalar and vector potentials Φ and $\boldsymbol{A}$ as

$$\tilde{\boldsymbol{E}}=-\nabla\tilde{\Phi}+\frac{\mathrm{i}\omega}{c}\tilde{\boldsymbol{A}}\qquad\tilde{\boldsymbol{B}}=\nabla\times\tilde{\boldsymbol{A}}. \tag{4}$$

In the low-beta limit we may take $\tilde{\boldsymbol{A}}\simeq\tilde{A}_\parallel\boldsymbol{B}_0/B_0$, assume $\tilde{B}_\parallel\simeq 0$, since $\tilde{B}_\parallel$ is related to the high-frequency compressional Alfvén branch, and drop terms involving the equilibrium pressure. Furthermore, since we are considering MHD-type modes, we assume $\tilde{E}_\parallel\simeq 0$, so that

$$\tilde{A}_\parallel\simeq\frac{c}{\mathrm{i}\omega}\left(\frac{\boldsymbol{B}_0}{B_0}\cdot\nabla\right)\tilde{\Phi}. \tag{5}$$

Now, with the use of equations (2)–(5), we obtain

$$\tilde{\boldsymbol{j}}_\perp=\frac{\mathrm{i}\omega c^2}{4\pi v_\mathrm{A}^2}\nabla_\perp\tilde{\Phi}+j_\parallel^0\frac{\tilde{\boldsymbol{B}}_\perp}{B_0} \tag{6}$$

where $v_\mathrm{A}=B_0/\sqrt{4\pi\rho_0}$ is the Alfvén velocity. The perturbed magnetic field $\tilde{\boldsymbol{B}}_\perp$ can approximately be expressed as

$$\tilde{\boldsymbol{B}}_\perp\simeq\nabla\left(\frac{\tilde{A}_\parallel}{B_0}\right)\times\boldsymbol{B}_0. \tag{7}$$

Using Amperè's law $\nabla\times\tilde{\boldsymbol{B}}=(4\pi/c)\tilde{\boldsymbol{j}}$ and equation (7), we can express the parallel perturbed current density as

$$\tilde{j}_\parallel=\frac{\boldsymbol{B}_0\cdot\tilde{\boldsymbol{j}}}{B_0}\simeq-\frac{c}{4\pi B_0}\nabla\cdot\left[B_0^2\nabla_\perp\left(\frac{\tilde{A}_\parallel}{B_0}\right)\right]. \tag{8}$$

Substituting equations (6)–(8) together with relation (5) into equation (1) yields the eigenmode equation, cf [32],

$$\frac{4\pi}{c}\left[\boldsymbol{B}_0\times\nabla\left(\frac{\boldsymbol{B}_0\cdot\nabla\tilde{\Phi}}{B_0^2}\right)\right]\cdot\nabla\left(\frac{j_\parallel^0}{B_0}\right)+(\boldsymbol{B}_0\cdot\nabla)\left(\frac{1}{B_0^2}\nabla\cdot\left[B_0^2\nabla_\perp\left(\frac{\boldsymbol{B}_0\cdot\nabla\tilde{\Phi}}{B_0^2}\right)\right]\right)$$
$$+\nabla\cdot\left(\frac{\omega^2}{v_\mathrm{A}^2}\nabla_\perp\tilde{\Phi}\right)=\frac{\mathrm{i}4\pi\omega}{c^2}\nabla\cdot\tilde{\boldsymbol{j}}^k. \tag{9}$$

The equilibrium parallel current density, $j_\parallel^0$, which appears in the second term in equation (9) is given by

$$j_\parallel^0 = \frac{\boldsymbol{B}_0 \cdot \boldsymbol{j}_0}{B_0} \simeq \frac{c}{4\pi} R\nabla \cdot \left(\frac{1}{R^2}\nabla\psi\right) \tag{10}$$

where R is the major radius and ψ is the poloidal flux. Except for the approximation of low-beta, equation (9) describes shear Alfvén waves in general axisymmetric geometry. However, for the sake of simplicity, it is convenient to consider equation (9) in the limit of small inverse aspect ratio, $\epsilon_a = a/R \ll 1$, where a is the minor radius at the plasma edge and R_0 is the major radius at the magnetic axis. Then, introducing the (r, θ, φ) coordinate system, where r is the minor radius coordinate, θ is the poloidal angle coordinate and φ is the toroidal angle coordinate, the equilibrium magnetic field can be represented as

$$\boldsymbol{B}_0 = \frac{rI}{Rq(r)}\nabla r \times \nabla(q\theta - \varphi) \tag{11}$$

where $I = B_\mathrm{T} R$, B_T is the toroidal magnetic field at the magnetic axis, $q = (\boldsymbol{B}_0 \cdot \nabla\varphi)/(\boldsymbol{B}_0 \cdot \nabla\theta)$ is the safety factor, $R = R_0 + r\cos\theta - \Delta(r) + r(r/R + \Delta')(\cos 2\theta - 1)/2$, with $\Delta(r)$ being the Shafranov shift of the flux surface, and $\Delta' \equiv \mathrm{d}\Delta/\mathrm{d}r$ is $O(\epsilon_a)$.

By virtue of periodicity in the poloidal and toroidal directions a wave field perturbation can be described by a Fourier decomposition in poloidal harmonics as

$$\tilde{\Phi} = \sum_m \Phi_m \mathrm{e}^{-\mathrm{i}m\theta + \mathrm{i}n\varphi - \mathrm{i}\omega t} \tag{12}$$

where n is the toroidal number. Since the equilibrium quantities are functions of θ, a coupling of the poloidal harmonics arises. To lowest order in the basic expansion parameter $\epsilon = r/R_0 \ll 1$, Φ_m couples only to its neighbouring sidebands $\Phi_{m\pm 1}$ and the couplings become significant in those terms of the eigenmode equation that involve a double derivative of $\tilde{\Phi}$ with respect to the radius. Then equation (9) reduces to a system of coupled equations given by, cf [6, 8, 32],

$$\hat{L}_{ij}(\omega)X_j = [\hat{L}_{ij}^{(0)}(\omega) + \hat{L}_{ij}^{(1)}(\omega)]X_j = 0 \tag{13}$$

where $X_i = \begin{pmatrix} \tilde{\Phi}_m \\ \tilde{\Phi}_{m+1} \end{pmatrix}$,

$$L_{ij}^{(0)} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}r} r\left(\frac{\omega^2}{v_\mathrm{A}^2} - k_{\parallel m}^2\right)\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}k_{\parallel m}^2}{\mathrm{d}r} - \frac{m^2}{r}\left(\frac{\omega^2}{v_\mathrm{A}^2} - k_{\parallel m}^2\right) & r\hat{\epsilon}\frac{\omega^2}{v_\mathrm{A}^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} \\ r\hat{\epsilon}\frac{\omega^2}{v_\mathrm{A}^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} & \frac{\mathrm{d}}{\mathrm{d}r} r\left(\frac{\omega^2}{v_\mathrm{A}^2} - k_{\parallel m+1}^2\right)\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}k_{\parallel m+1}^2}{\mathrm{d}r} - \frac{(m+1)^2}{r}\left(\frac{\omega^2}{v_\mathrm{A}^2} - k_{\parallel m+1}^2\right) \end{pmatrix} \tag{14}$$

and

$$L_{ij}^{(1)}(\omega)X_j = -\frac{4\pi\mathrm{i}\omega}{c^2}\begin{pmatrix} \frac{\partial}{\partial r}(rj_{r,m}^k) - \mathrm{i}mj_{\theta,m}^k \\ \frac{\partial}{\partial r}(rj_{r,m+1}^k) - \mathrm{i}(m+1)j_{\theta,m+1}^k \end{pmatrix}. \tag{15}$$

Here, the subscripts m and $m+1$ are the dominant poloidal mode numbers, $k_{\parallel m} = (nq - m)/qR_0$ is the parallel wave number and $\hat{\epsilon} = dr/R_0$ is the coupling quantity with $d = 2.5$–5. The value $d = 2.5$ is obtained by approximating Δ' with its value corresponding to a plasma with zero beta and constant current profile, namely $\Delta' = r/4R_0$ [32].

Note that equation (15) represents the kinetic response due to the wave–particle interaction. In the following, we shall concentrate on the kinetic effects of the energetic ion populations on the TAE modes. Assuming the kinetic part of equation (13) to be small we may apply the method of perturbations for finding the eigenvalues and the eigenfunctions. In the zero approximation the kinetic part is neglected and the two coupled second-order eigenmode equations describe the TAE eigenmode, which is a shear

Alfvén wave whose frequency lies within a 'gap' in the frequency spectrum that is induced by finite toroidicity. Specifically, the gap appears at the points $r = r_m$ where $k_{\|m+1}(r_m) = -k_{\|m}(r_m) = 1/(2q_m R_0)$, with $q_m = q(r_m) = (2m+1)/2n$ and the local frequency in the gap is $\omega = v_A(r_m)/(2q_m R_0)$, cf [3–5]. Now the inclusion of the resonant interaction between an energetic ion population and the TAE mode, represented by the operator $\hat{L}^{(1)}_{ij}(\omega)$, may lead to a strong destabilization of the mode [6–17].

The instability growth rate can be calculated perturbatively by assuming that the imaginary part of the frequency is small compared with the real part, $\omega = \omega_0 + \mathrm{i}\gamma$, $|\gamma| < \omega_0$. Expanding also the eigenfunctions as $X_i = X^{(0)}_i + X^{(1)}_i$, where $\hat{L}^{(0)}_{ij} X^{(0)}_j = 0$, $|X^{(1)}_i / X^{(0)}_i| \ll 1$ and taking into account the self-adjointness of the operator $L^{(0)}_{ij}$, we obtain from equation (13) the imaginary part of the frequency as

$$\gamma = \mathrm{i}\frac{\displaystyle\int_0^a r\,\mathrm{d}r\,X^{(0)}_i \overline{L^{(1)}_{ij} X^{(0)}_j}}{\displaystyle\int_0^a r\,\mathrm{d}r\,X^{(0)}_i \frac{\partial L^{(0)}_{ij}}{\partial\omega} X^{(0)}_j} \tag{16}$$

where the bar over the integrand of the nominator means the flux-surface (poloidal) averaging, $\overline{(\ldots)} \equiv \int (h/2\pi)\,\mathrm{d}\theta(\ldots)$ with $h = 1 + \epsilon\cos\theta$.

In the next section, we are going to derive the energetic ion kinetic response and find a general analytic estimate of the energetic particle-induced TAE growth rate.

3. Energetic particle response

The energetic particle kinetic response enters the eigenmode equation through the current density perturbation, as may be seen in equation (16). Since

$$\tilde{\boldsymbol{j}}^k = e\int \mathrm{d}^3v\,\boldsymbol{v}\,\tilde{f} \tag{17}$$

where e is the charge of the fast ions, it is necessary to determine the perturbed part of the velocity distribution function of the fast ions, $\tilde{f}$. A well known approach for obtaining $\tilde{f}$ is to solve the linearized kinetic Vlasov equation by applying the method of characteristics. However, we present here an alternative and very convenient solution method which is based on employing the Hamiltonian action-angle variables introduced by Kaufman [34]. The Vlasov equation describing the distribution function of the energetic particles interacting with a (weak) wave field can be written in the form

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial\boldsymbol{\alpha}}(\dot{\boldsymbol{\alpha}} f) + \frac{\partial}{\partial\boldsymbol{J}}(\dot{\boldsymbol{J}} f) = \frac{\partial f}{\partial t} + \dot{\boldsymbol{\alpha}}\frac{\partial f}{\partial\alpha} + \dot{\boldsymbol{J}}\frac{\partial f}{\partial\boldsymbol{J}} = 0 \tag{18}$$

where the Hamiltonian of the system $H = H(\boldsymbol{J}, \boldsymbol{\alpha}, t)$ relates the action variables $\boldsymbol{J}$ and the angles $\boldsymbol{\alpha}$ through the Hamilton equations

$$\frac{\partial H}{\partial\boldsymbol{\alpha}} = -\dot{\boldsymbol{J}} \qquad \text{and} \qquad \frac{\partial H}{\partial\boldsymbol{J}} = \dot{\boldsymbol{\alpha}}. \tag{19}$$

In the presence of weak electromagnetic fields the Hamiltonian can be expanded as

$$H(\boldsymbol{J}, \boldsymbol{\alpha}, t) = H_0(\boldsymbol{J}) + \tilde{H}(\boldsymbol{J}, \boldsymbol{\alpha}, t) \qquad |\tilde{H}| \ll H_0 \tag{20}$$

where H_0 corresponds to the unperturbed particle motion, while $\tilde{H}$ is the perturbation due to the electromagnetic waves. Thus, J^i are the invariants of the unperturbed particle motion. Expanding then the distribution function, f, as

$$f(\boldsymbol{J}, \boldsymbol{\alpha}, t) = f_0(\boldsymbol{J}) + \tilde{f}(\boldsymbol{J}, \boldsymbol{\alpha}, t) \qquad |\tilde{f}| \ll f_0 \tag{21}$$

where f_0 is the unperturbed part of f, we find from equation (18) to first order that

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\Omega}\frac{\partial}{\partial \boldsymbol{\alpha}}\right)\tilde{f} = -\dot{\boldsymbol{J}}\frac{\partial f_0}{\partial \boldsymbol{J}} \tag{22}$$

where $\boldsymbol{\Omega} = \partial\boldsymbol{\alpha}/\partial t = \partial H/\partial \boldsymbol{J}$ and $\partial \boldsymbol{J}/\partial t = -\partial H/\partial \boldsymbol{\alpha}$. A Fourier decomposition in $\boldsymbol{\alpha}$ and t of the perturbed quantities

$$\tilde{f}(\boldsymbol{J}, \boldsymbol{\alpha}, t) = \sum_{\boldsymbol{n}} \tilde{f}(\boldsymbol{J}, \boldsymbol{n}, \omega)\mathrm{e}^{\mathrm{i}(\boldsymbol{n}\cdot\boldsymbol{\alpha}-\omega t)} \tag{23}$$

$$\tilde{H}(\boldsymbol{J}, \boldsymbol{\alpha}, t) = \sum_{\boldsymbol{n}} \tilde{H}(\boldsymbol{J}, \boldsymbol{n}, \omega)\mathrm{e}^{\mathrm{i}(\boldsymbol{n}\cdot\boldsymbol{\alpha}-\omega t)} \tag{24}$$

where the summation is taken over the three integers $\boldsymbol{n} = (n^1, n^2, n^3)$, then gives

$$\frac{\partial \boldsymbol{J}}{\partial t} = -\frac{\partial H}{\partial \boldsymbol{\alpha}} = -\sum_{\boldsymbol{n}} \mathrm{i}\boldsymbol{n}\tilde{H}(\boldsymbol{J}, \boldsymbol{n}, \omega)\mathrm{e}^{\mathrm{i}(\boldsymbol{n}\cdot\boldsymbol{\alpha}-\omega t)} \tag{25}$$

and

$$\tilde{f}(\boldsymbol{J}, \boldsymbol{n}, \omega) = \tilde{H}(\boldsymbol{J}, \boldsymbol{n}, \omega)\frac{\boldsymbol{n}}{(\boldsymbol{n}\cdot\boldsymbol{\Omega} - \omega)} \cdot \frac{\partial f_0}{\partial \boldsymbol{J}}. \tag{26}$$

According to [34] the action variables for the guiding centre motion are

$$J^1 = cM\mu/e \qquad J^2 = \frac{1}{2\pi}\int p_\theta\,\mathrm{d}\theta \qquad J^3 = Mb_\varphi R v_\parallel - \frac{e}{c}\Psi_\varphi \tag{27}$$

where M is the particle mass, μ is the magnetic moment, $b_\varphi = R\nabla\varphi\cdot\boldsymbol{B}_0/B_0$, $v_\parallel = \boldsymbol{B}_0\cdot\boldsymbol{v}/\boldsymbol{B}_0$ and p_θ is the poloidal canonical angular momentum. The conjugated angles are such that α^1 describes the position of the particle in the Larmor rotation, α^2 the position along the guiding centre orbit and α^3 the toroidal position of the banana centre. The frequencies Ω^i are $\Omega^1 = \langle\omega_\mathrm{c}\rangle$, the bounce averaged cyclotron frequency with $\omega_\mathrm{c} = eB_0/cM$, $\Omega^2 = \omega_\mathrm{b}$, the bounce frequency, and $\Omega^3 = \langle\dot{\varphi}\rangle$, the precessional drift frequency, respectively.

Because the expression for J^2 is complicated (it involves an integral over the orbit), we use instead of $\boldsymbol{J}$ the set of invariants $\boldsymbol{I} = (\mathcal{E}, \mu, J_\varphi)$, where $\mathcal{E} = Mv^2/2 = H_0$ is the particle energy, $J_\varphi = -J^3/MR_0\omega_\mathrm{c} = \psi - hv_\parallel/\omega_\mathrm{c}$ is the toroidal canonical angular momentum with $\psi = \psi_\mathrm{p}/R_0B_0$, $b_\varphi \simeq 1$ and $h = 1 + \epsilon\cos\theta$. Then equation (26) becomes

$$\tilde{f}(\boldsymbol{I}, \boldsymbol{n}, \omega) = \frac{\tilde{H}(\boldsymbol{I}, \boldsymbol{n}, \omega)}{\boldsymbol{n}\cdot\boldsymbol{\Omega} - \omega}\boldsymbol{n}\cdot\left(\boldsymbol{\Omega}\frac{\partial}{\partial\mathcal{E}} + \delta_{i,1}\frac{e}{cM}\frac{\partial}{\partial\mu} - \delta_{i,3}\frac{1}{M\omega_\mathrm{c}R_0}\frac{\partial}{\partial J_\varphi}\right)f_0. \tag{28}$$

The perturbed part of the Hamiltonian

$$H(\boldsymbol{p}, \boldsymbol{\alpha}, t) = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + e\Phi(\boldsymbol{\alpha}, t) \tag{29}$$

is given by

$$\tilde{H} = -\frac{e}{c}\tilde{\boldsymbol{A}}\cdot\boldsymbol{v} + e\tilde{\Phi} = \tilde{H}_\omega\mathrm{e}^{-\mathrm{i}\omega t} = \frac{\mathrm{i}e}{\omega}\boldsymbol{v}\cdot\tilde{\boldsymbol{E}}_\omega(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega t} \tag{30}$$

where $\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_\omega\mathrm{e}^{-\mathrm{i}\omega t}$. Note that $\boldsymbol{r}$ and $\boldsymbol{v}$ here are the particle (not guiding centre) variables. If the scale length for the variation of $\boldsymbol{E}_\omega$ is larger than the Larmor radius, we can write the particle position as a sum of that of the guiding centre and that pertaining to the Larmor rotation, $\boldsymbol{r} = \boldsymbol{r}_\mathrm{L} + \boldsymbol{r}_\mathrm{g}$ and then use the approximation $\tilde{\boldsymbol{E}}_\omega(\boldsymbol{r}) \simeq \tilde{\boldsymbol{E}}_\omega(\boldsymbol{g}r_\mathrm{g})\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{r}_\mathrm{L}}$, where $\boldsymbol{k}_\perp$ is the perpendicular component of the wavevector. Now, taking $\boldsymbol{k}_\perp \simeq k_\theta\hat{\theta}$ and $\boldsymbol{r}_\mathrm{L} = \frac{v_\perp}{\omega_\mathrm{c}}(-\hat{r}\sin\delta + \hat{\theta}\cos\delta)$, where δ is the gyrophase we can write expression (30) as

$$\tilde{H}_\omega = \frac{\mathrm{i}e}{\omega}(\boldsymbol{v}_\mathrm{D} + \boldsymbol{v}_\mathrm{L})\cdot\tilde{\boldsymbol{E}}_\omega(\boldsymbol{r}_\mathrm{g})\mathrm{e}^{-\mathrm{i}\xi\sin\delta} = \frac{\mathrm{i}e}{\omega}\sum_l \tilde{\boldsymbol{E}}_\omega(\boldsymbol{r}_\mathrm{g})\cdot\boldsymbol{Q}_l\mathrm{e}^{-\mathrm{i}l\delta}. \tag{31}$$

Here $\xi = k_\theta v_\perp/\omega_c$ and $\boldsymbol{v}_L = v_\perp(\hat{r}\cos\delta + \hat{\theta}\sin\delta)$ is the velocity of the Larmor rotation. Furthermore, we have assumed that $\tilde{E}_{\omega\|} \simeq 0$ and the drift velocity $\boldsymbol{v}_D = \mathrm{d}\boldsymbol{r}_{g\perp}/\mathrm{d}t$ is given by $\boldsymbol{v}_D = -v_D(\hat{r}\sin\theta + \hat{\theta}\cos\theta)$, with $v_D = (v_\|^2 + v_\perp^2/2)/R_0\omega_c$ and

$$\boldsymbol{Q}_l = \left\{ \left(v_{Dr} + v_\perp \frac{l}{\xi} \right) J_l(\xi),\, (v_{D\theta} + \mathrm{i} v_\perp \frac{\mathrm{d}}{\mathrm{d}\xi}) J_l(\xi) \right\} \tag{32}$$

where $J_l(\xi)$ is the Bessel function of the first kind. Next we decompose $\tilde{\boldsymbol{E}}_\omega(\boldsymbol{r}_g)$ with respect to the poloidal and toroidal angles according to equation (12) and Fourier transform $\tilde{H}_\omega$ with respect to $\boldsymbol{\alpha}$:

$$\tilde{H}(\boldsymbol{I}, \boldsymbol{n}, \omega) = \int \frac{\mathrm{d}\boldsymbol{\alpha}}{(2\pi)^3} \tilde{H}_\omega \mathrm{e}^{-\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{\alpha}} = \sum_l \int \frac{\mathrm{d}\alpha}{(2\pi)^3} \frac{\mathrm{i}e}{\omega} \tilde{\boldsymbol{E}}_m(r) \cdot \boldsymbol{Q}_l \mathrm{e}^{-\mathrm{i}(l\delta + \boldsymbol{n}\cdot\boldsymbol{\alpha} + m\theta - n\varphi)} \tag{33}$$

where r, θ, φ and δ are to be considered as functions of $\boldsymbol{\alpha}$. Since r, θ and φ describe the position of the guiding centre they are independent of α^1. Moreover, θ does not depend on α^3 because of axisymmetry. The gyroangle δ evolves at the local cyclotron frequency $\dot{\delta} = -\omega_c(\boldsymbol{r}_g) \equiv -\Omega^1 - \tilde{\omega}_c(\boldsymbol{I}, \alpha^2)$, where $\tilde{\omega_c}$ is the oscillating part of $\omega_c(\boldsymbol{r}_g)$ and is of the order of $\epsilon\Omega^1$, and thus $\delta + \alpha^1$ depends only on α^2. For these reasons, the integral in (33) vanishes unless $n^1 = l$ and $n^3 = n$, and the integrations over α^1 and α^3 are trivial. Changing the integration variable to $t = \alpha^2/\Omega^2$, we thus obtain

$$\tilde{H}(\boldsymbol{I}, \boldsymbol{n}, \omega) = \frac{\mathrm{i}e}{\omega} \sum_{l,m} \int_0^{\tau_b} \frac{\mathrm{d}t}{\tau_b} \tilde{\boldsymbol{E}}_m(r) \cdot \boldsymbol{Q}_l \exp\left(\mathrm{i} \int_0^t \mathrm{d}t'(-m\dot{\theta} + n\dot{\varphi} + l\omega_c - \boldsymbol{n}\cdot\boldsymbol{\Omega}) \right) \tag{34}$$

where $\tau_b = 2\pi/\Omega^2$. Substituting (28) together with (34) into (17), using Landau's rule $(\boldsymbol{n}\cdot\boldsymbol{\Omega} - \omega)^{-1} = -\pi\mathrm{i}\delta(\boldsymbol{n}\cdot\boldsymbol{\Omega} - \omega)$ to retain the resonant part of the kinetic integral and applying the relation

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\delta (\boldsymbol{v}_g + \boldsymbol{v}_L) \mathrm{e}^{\mathrm{i}(\xi\sin\delta - l\delta)} = \boldsymbol{Q}_l^\star \tag{35}$$

we arrive after some algebra at

$$\begin{aligned} \boldsymbol{j}_m^k(r) = -\sum_{s,m} \frac{\pi e^2}{M^2\omega} \int \mathcal{E}\,\mathrm{d}\mathcal{E} \int \frac{\mathrm{d}\lambda}{|v_\||} \boldsymbol{Q}_l^\star \mathrm{e}^{\mathrm{i}S(0,t)} \delta(\omega - l\langle\omega_c\rangle + m\langle\dot{\theta}\rangle - n\langle\dot{\varphi}\rangle - s\omega_b) \\ \cdot\hat{\Pi}(\boldsymbol{n}) f_0 \int_0^{\tau_b} \frac{\mathrm{d}t}{\tau_b} \boldsymbol{Q}_l \cdot \boldsymbol{E}_m(r) \mathrm{e}^{-\mathrm{i}S(0,t)} \end{aligned} \tag{36}$$

where $\langle\ldots\rangle = \int(\mathrm{d}t/\tau_b)(\ldots)$, $\lambda = B_0\mu/\mathcal{E}$, $\sigma = \mathrm{sgn} v_\|$, $n^2 = s - m$ and

$$\begin{aligned} \hat{\Pi} &\equiv \omega\frac{\partial}{\partial\mathcal{E}} + \frac{l\omega_c}{B_0}\frac{\partial}{\partial\mu} - \frac{n}{M\omega_c R}\frac{\partial}{\partial J_\varphi} \\ S_l(0,t) &= \int_0^{\mathrm{t}} \mathrm{d}t'(\omega - l\omega_c + m\dot{\theta} - n\dot{\varphi}). \end{aligned} \tag{37}$$

Since $\omega \ll \langle\omega_c\rangle$ for the considered TAE mode, we take $l = 0$ in the above expressions. Finally, we use the relation

$$r\,\mathrm{d}r h\,\mathrm{d}\theta = |v_\|| \,\mathrm{d}t\,\mathrm{d}J_\varphi$$

and obtain from equations (14)–(16) and (36) the following expression for γ

$$\gamma = -\frac{2\pi}{c^2} \frac{\sum_m \int r\,\mathrm{d}r \oint (h/2\pi)\,\mathrm{d}\theta\, \boldsymbol{E}_m^\star \cdot \boldsymbol{j}_m^k}{\sum_m \int (r/v_A^2)\,\mathrm{d}r |\tilde{\boldsymbol{E}}_m|^2} \tag{38}$$

$$= \frac{4\pi^3 e^2}{c^2\omega_r M^2} \frac{\sum_{s,m,\sigma}\int \mathrm{d}J_\varphi \int E\,\mathrm{d}E \int (1/\omega_b)\mathrm{d}\lambda\,\delta(\omega_r + m\langle\dot{\theta}\rangle - n\langle\dot{\varphi}\rangle - s\omega_b)|G_m|^2 \hat{\Pi} f_0}{\sum_m \int \mathrm{d}r\, r(|\tilde{\Phi}'_m|^2 + (m^2/r^2)|\tilde{\Phi}_m|^2) v_A^{-2}}$$

where

$$G_m = \int_0^{\tau_b} \frac{\mathrm{d}t}{\tau_b} v_D \left[\tilde{\Phi}'_m \sin\theta - \frac{\mathrm{i}m}{r}\tilde{\Phi}_m \left(\cos\theta - \mathrm{i}\frac{v_\perp}{v_D}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\right] J_0(\xi)\mathrm{e}^{-\mathrm{i}S_0(0,t)} \quad (39)$$

and $\hat{\Pi}$ is given by equation (37) with $l = 0$. $\dot{\theta} \equiv \mathrm{d}\theta/\mathrm{d}t$ and $\dot{\varphi} \equiv \mathrm{d}\varphi/\mathrm{d}t$ are determined by the particle orbital motion. The operator $\hat{\Pi}$ shows that the instability drive is proportional to the toroidal wavenumber rather than the poloidal one. This is in agreement with the scaling obtained in [10] (see also [35, 36]) but in disagreement with [9, 11, 12, 14].

The above expression for γ, is appropriate for description of the TAE destabilization by both circulating and trapped fast ions. It takes into account large particle orbit width effects, the effect of mode localization as well as the finite Larmor orbit (FLR) effects of energetic ions. The influence of these effects on the TAE instability growth rate is analysed in the next section.

4. TAE growth rate

The destabilization of the TAE modes by both passing and trapped ions is sensitive to the ratio between the radial excursion of the particle orbits from the flux surfaces, Δ_b, and the TAE localization scale length, Δ_m, cf [13, 14, 16, 17]. According to [17], in the limit of low shear and large mode numbers, the structure of the TAE mode can be determined within a 'single-gap' approximation by solving the eigenmode equation system inside and outside the gap separately and then matching the solutions at the gap surfaces. The corresponding inner and outer structures are described by

$$\tilde{\Phi}_m^{(i)} = C_m \left[\frac{1}{\alpha_m}\arctan\left(\frac{r - r_m}{\Delta_m^{(i)}}\right) + \frac{1}{2}\ln\left|\left(\frac{r - r_m}{\Delta_m^{(i)}}\right)^2 + 1\right|\right] + \text{constant} \quad (40)$$

$$\tilde{\Phi}_m^{(o)} = -C_m K_0\left(\left|\frac{r - r_m}{\Delta_m^{(o)}}\right|\right) \quad (41)$$

where $K_0(x)$ is the zeroth-order MacDonald function, $\hat{s} = r_m q'(r_m)/q(r_m)$ is the magnetic shear, $\alpha_m = 4/(\pi\hat{s})$, r_m is the mode localization radius, $\Delta_m^{(i)} = (\pi/8)\mathrm{d}r_m^2/mR_0$ with $d = 2.5$–5 and $\Delta_m^{(o)} = r_m/m$ are the inner and outer widths of the mode, respectively, and C_m is a constant determined by the boundary conditions at the gap surface. In spite of the fact that $\Delta_m^{(i)}/\Delta_m^{(o)} = r_m/R_m \ll 1$ most of the mode energy is concentrated in the inner region of the mode. Estimating Δ_b for the well-passing particles as $\Delta_b^p \simeq qv/\omega_c$ and $v \simeq v_A$ one finds that $\Delta_b^p/\Delta_m^{(i)} \simeq (L_\alpha/r_m)(\omega_{\star\alpha}/\omega_r)$, where $\omega_{\star\alpha} \simeq nqv^2/(\omega_c r_m L_\alpha)$ is the diamagnetic drift frequency and L_α is the scale length of the particle density gradient. Since $\omega_\star/\omega_r > 1$ is required for the instability, [6], and $L_\alpha \sim r_m$, it follows that $\Delta_b^p/\Delta_m^{(i)} > 1$ can be fulfilled. In the case of excitation by trapped particles the width of the banana orbits is $(\Delta_b^t)_{max} \simeq 2\sqrt{2}q_m v/(\omega_c\epsilon_m^{1/2})$ and the condition $(\Delta_b^t)_{max}/\Delta_m^{(i)} > 1$ becomes even easier to satisfy than for passing particles. Furthermore, when most of the energetic particles are 'born' in the central plasma region as in the case of the fusion-produced alpha particles the excursion of particle orbits can be comparable to or larger than the orbit width of the excited

TAE mode, i.e. $\Delta_b \geqslant \Delta_m^{(o)}$. This takes place for passing particles if $qv_A/\omega_c \geqslant r_m/m$ and for trapped particles if $qv_A/\omega_c \geqslant \sqrt{\epsilon_m} r_m/m$.

In order to clearly demonstrate the influence of finite particle orbit effects on the TAE instability we will compare the growth rate expression (38) for passing and trapped particles in the two opposite limits: $\Delta_b \ll \Delta_m^{(i)}$ (local theory) and $\Delta_b \gg \Delta_m^{(i)}$ (nonlocal theory). In the local approximation we will assume that the growth rate is insensitive to the mode structure and that $|\tilde{\Phi}'_m(r_m)| \gg |m\tilde{\Phi}_m(r_m)/r|$, where the prime denotes the radial derivative. In the nonlocal theory two cases will be considered: $\Delta_m^{(i)} \ll \Delta_b \ll \Delta_m^{(o)}$ and $\Delta_b \geqslant \Delta_m^{(o)}$, where the particle drive is mainly determined by the inner and outer mode structure, respectively. Note that the growth rate expression includes both the instability drive and the Landau damping due to the energetic ions. However, in order to determine the instability threshold it is also necessary to take into account contributions from other various damping mechanisms, [8–11, 30–33]. This problem is beyond the scope of the present work.

4.1. Local theory: $\Delta_b \ll \Delta_m^{(i)}$

4.1.1. Passing particles. Generally, the main difficulty in calculating the growth rate explicitly is the integration along the particle drift orbits in equation (39). Using the equations for particle drift motion

$$\dot{r} = -v_D \sin\theta \tag{42}$$

$$\dot{\theta} = \frac{v_\parallel}{qR} - \frac{v_D}{r}\cos\theta \tag{43}$$

$$\dot{\varphi} = \frac{v_\parallel}{R} \tag{44}$$

the function G_m in equation (39) can be rewritten as

$$G_m = -\mathrm{i}\int_0^{\tau_b} \frac{\mathrm{d}t}{\tau_b^p}\tilde{\Phi}_m(r)\left(\omega - k_{\parallel m}v_\parallel - \frac{\mathrm{i}mv_\perp}{r}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)J_0(\xi)\mathrm{e}^{-\mathrm{i}S_0(0,t)}. \tag{45}$$

For well-passing particles we find from equations (42)–(44) that $\theta + \pi \simeq \omega_b^p t \simeq (v_\parallel/qR)t$ and $r = r_0 + \Delta_b^p \cos\theta$, where $\Delta_b^p = v_D/\omega_b^p$. Taking into account the resonance condition

$$\omega_r - k_{\parallel m}v_\parallel - s\omega_b^p = 0 \tag{46}$$

and neglecting the FLR effect represented by the term $\mathrm{d}J_0(\xi)/\mathrm{d}\xi$ in equation (45) we obtain

$$G_m \simeq -\frac{\mathrm{i}s}{\tau_b^p}\int_{-\pi}^{\pi}\mathrm{d}\theta\,\tilde{\Phi}_m(r_0 + \Delta_b^p\cos\theta)J_0(\xi)\mathrm{e}^{-\mathrm{i}s(\pi+\theta)+\mathrm{i}(\Delta_b^p m/r_0)\sin\theta}$$

$$= -\mathrm{i}\frac{s(-1)^s}{\tau_b^p}J_0(\xi)\sum_{p=-\infty}^{+\infty}J_p(\eta)\int_{-\pi}^{\pi}\mathrm{d}\theta\,\tilde{\Phi}_m(r_0 + \Delta_b^p\cos\theta)\mathrm{e}^{\mathrm{i}(p-s)\theta} \tag{47}$$

where $\eta = \Delta_b^p m/r_0$. In the limit $\Delta_b^p \ll \Delta_m^{(i)}$ the function G_m is determined by the inner mode structure. Since $\eta \ll 1$, the expansion of $J_p(\eta)$ and $\tilde{\Phi}_m^{(i)}(r_0 + \Delta_b^p\cos\theta)$ to $O(\eta)$ yields

$$G_m \simeq \frac{\mathrm{i}s\omega_b}{2}[\eta(\delta_{s,1} - \delta_{s,-1})\tilde{\Phi}_m^{(i)}(r_0) + \Delta_b^p(\delta_{s,1} + \delta_{s,-1})\tilde{\Phi}_m^{(i)'}(r_0)]J_0(\xi) \tag{48}$$

where $\delta_{s,p}$ is the Kronecker delta. Here, we use the fact that $|\tilde{\Phi}_m^{(i)'}(r_0)/\tilde{\Phi}_m^{(i)}(r_0)| \sim 1/\Delta_m^{(i)} \gg 1/\Delta_m^{(o)} \sim \eta/\Delta_b^p$ which implies that the first term in equation (48) can be neglected. This gives

$$|G_m|^2 \simeq \frac{v_D^2}{4}|\tilde{\Phi}_m^{(i)'}|^2 J_0^2(\xi)(\delta_{s,1} + \delta_{s,-1}). \tag{49}$$

Substituting (49) into (38) and changing the integration variables according to $\mathrm{d}J_\varphi \simeq \mathrm{d}r\, r/Rq$ and $\sum_\sigma \mathrm{d}v\, v^3\, \mathrm{d}\lambda = 2v_\parallel v_\perp\, \mathrm{d}v_\parallel\, \mathrm{d}v_\perp$ we obtain the following expression for the growth rate,

$$\frac{\gamma_\mathrm{p}^\mathrm{l}}{\omega_\mathrm{r}} = \frac{8\pi^3 q^3 M n}{B_0^2 \omega_\mathrm{c}} \sum_{v_\mathrm{r}=v_\mathrm{A}, v_\mathrm{A}/3} \int_0^{v_\mathrm{r}/\sqrt{2\epsilon}} v_\perp\, \mathrm{d}v_\perp\, J_0^2(\xi) \frac{v_\mathrm{r}}{v_\mathrm{A}} \left(v_\mathrm{r}^2 + \frac{v_\perp^2}{2} \right)^2 \left(\frac{\omega_\mathrm{r}}{\omega_\star^\mathrm{p}} - 1 \right) \frac{\partial f_0}{\partial J_\varphi} \bigg|_{\substack{v_\parallel = v_\mathrm{r} \\ J_\varphi = \Psi_m}} \tag{50}$$

where

$$\xi = \frac{m v_\perp}{r \omega_\mathrm{c}} \qquad \omega_\star^\mathrm{p} = \frac{n v_\parallel}{\omega_\mathrm{c} R_0} \frac{\partial f_0 / \partial J_\varphi}{\partial f_0 / \partial v_\parallel} \qquad \omega_\mathrm{r} = v_\mathrm{A}/2qR$$

and the superscript ‘l’ and the subscript ‘p’ refer to local theory and passing particles, respectively. The summation over v_r in (50) corresponds to the possible resonances $|v_\parallel| = v_\mathrm{A}$ and $|v_\parallel| = v_\mathrm{A}/3$, and all quantities are calculated for $r = r_m$. It is obvious from equation (50) that the FLR corrections have a small stabilizing effect if $\xi > 1$, cf [18]. When the FLR effect is neglected, i.e. when $J_0^2(\xi) \simeq 1$, the above expression for the growth rate reproduces the result which have been obtained for a Maxwellian distribution function of energetic particles [6, 12], as well as for a slowing-down distribution generated either by NBI, or by an isotropic source such as internally-produced alpha particles [12], if n is replaced by m/q in the expression for $\omega_\star^\mathrm{p}$.

4.1.2. Trapped particles. We consider here well-trapped particles with the trapping parameter $\kappa^2 \equiv (1 - \lambda + \lambda\epsilon)/(2\epsilon\lambda) \ll 1$ and assume narrow banana orbits, i.e. $|qv_\parallel/(\epsilon r \omega_\mathrm{c})| \ll 1$. With these assumptions and making use of the drift equations (42)–(44) we find that

$$S_0(0, t) \simeq qR_0 k_{\parallel m}(\theta - \theta_1) + (\omega - \omega_\mathrm{D})t \tag{51}$$

with $\theta_1 = -2 \arcsin \kappa$ and the relation between time and poloidal angle defined by

$$\omega_\mathrm{b} t = \sigma \left[\frac{\pi}{2} + \arcsin\left(\frac{1}{\kappa} \sin\frac{\theta}{2} \right) \right]$$

$\omega_\mathrm{b}^\mathrm{t} \simeq \sqrt{(\epsilon/2)}(v/qR_0)$ is the bounce frequency and $\omega_\mathrm{D}^\mathrm{t} \simeq nqv^2/2rR_0\omega_\mathrm{c}$ is the precessional drift frequency. The resonance condition becomes

$$\omega_\mathrm{r} - \omega_\mathrm{D}^\mathrm{t} - s\omega_\mathrm{b}^\mathrm{t} = 0. \tag{52}$$

Using the same arguments as in the case of passing particles we find that the function G_m is mainly determined by the radial derivative of the inner mode structure at $r = r_m$. Consequently we have

$$\begin{aligned} G_m &\simeq \tilde{\Phi}_m^{(\mathrm{i})\prime} v_\mathrm{D} J_0(\xi) \int_0^{\tau_\mathrm{b}^\mathrm{t}} \frac{\mathrm{d}t}{\tau_\mathrm{b}^\mathrm{t}} \sin\theta\, \mathrm{e}^{-\mathrm{i}S_0(0,t)} \\ &\simeq \tilde{\Phi}_m^{(\mathrm{i})\prime} \frac{\kappa v_\mathrm{D}}{\pi} J_0(\xi) \sum_\sigma \mathrm{e}^{-\frac{1}{2}\mathrm{i}(\theta_1 + s\pi\sigma)} \int_{-\pi/2}^{\pi/2} \mathrm{d}\alpha\, \sin\alpha\, \mathrm{e}^{\mathrm{i}(\kappa\sin\alpha - s\alpha\sigma)} \\ &\simeq -\mathrm{i}\tilde{\Phi}_m^{(\mathrm{i})\prime} 2\kappa v_\mathrm{D} J_0(\xi) J_0''(\kappa) \mathrm{e}^{-\frac{1}{2}\mathrm{i}\theta_1} \end{aligned} \tag{53}$$

where the relations $\alpha = \arcsin((1/\kappa)\sin(\theta/2))$, $k_{\parallel m} = 1/(2qR_0)$ and

$$\frac{\partial}{\partial\kappa}(\mathrm{e}^{\mathrm{i}\kappa\sin\alpha}) = \sum_p J_\mathrm{p}'(\kappa)\mathrm{e}^{\mathrm{i}p\alpha} \tag{54}$$

have been used. Retaining only the lowest-order term in the expansion of $J_0''(\kappa)$ and using $v_{\rm D} \simeq v^2/2\omega_{\rm c} R_0$, we obtain

$$|G_m|^2 = \frac{v^4}{4\omega_{\rm c}^2 R_0^2}\kappa^2|\tilde{\Phi}_m^{({\rm i})'}|^2 J_0^2(\xi). \tag{55}$$

The resonance condition determines the resonant velocity as

$$v_1 = -\sqrt{\frac{\epsilon}{2}}\frac{r\omega_{\rm c}}{nq^2} + \sqrt{\left(\sqrt{\frac{\epsilon}{2}}\frac{r\omega_{\rm c}}{nq^2}\right)^2 + \frac{r\omega_{\rm c} v_{\rm A}}{nq^2}}. \tag{56}$$

Now substituting (55) into (38) and transforming the integration variables according to $\mathrm{d}\lambda \simeq -2\epsilon\, \mathrm{d}\kappa^2$, we arrive at the following expression for the growth rate

$$\frac{\gamma_{\rm t}^{\rm l}}{\omega} = \frac{8\pi^3\sqrt{2\epsilon}q^2 nM}{B_0^2 R\omega_{\rm c}} J_0^2\left(\frac{mv_1}{r\omega_{\rm c}}\right)\frac{v_1^6}{D_1}\int_0^{\kappa_{\max}^2}\kappa^2\,\mathrm{d}\kappa^2\left(\frac{\omega_{\rm r}}{\omega_\star^{\rm t}} - 1\right)\frac{\partial f_0}{\partial J_\varphi}\bigg|_{\substack{v=v_1\\ J_\varphi=\Psi_m}} \tag{57}$$

where the diamagnetic drift frequency is

$$\omega_\star^{\rm t} = \frac{nv}{\omega_{\rm c} R_0}\frac{\partial f_0/\partial J_\varphi}{\partial f_0/\partial v}$$

all quantities are calculated at $r = r_m$, and

$$D_1 = \frac{nqv_1}{rR_0\omega_{\rm c}} + \frac{1}{qR_0}\sqrt{\frac{\epsilon}{2}}. \tag{58}$$

Equations (56)–(58) take into account the effect of simultaneous resonant interaction with the bounce motion and the toroidal precession motion of fast ions, which may significantly affect the magnitude of the resonant velocity and the growth rate. The influence of the precessional drift frequency on the resonance condition can be neglected if $v_{\rm A}/(\omega_{\rm c} r_m) \ll \epsilon_m/(nq^2)$.

4.2. Nonlocal theory: $\Delta_m^{({\rm i})} \ll \Delta_{\rm b} \ll \Delta_m^{({\rm o})}$

4.2.1. Passing particles. In the considered case with $\Delta_m^{({\rm i})} \ll \Delta_{\rm b}^{\rm p} \ll \Delta_m^{({\rm o})}$, the function G_m is still determined mainly by the radial derivative of the inner mode structure, i.e.

$$G_m \simeq \int_0^{\tau_{\rm b}^{\rm p}}\frac{\mathrm{d}t}{\tau_{\rm b}^{\rm p}}\tilde{\Phi}_m^{({\rm i})'}(r)v_{\rm D}\sin\theta\, \mathrm{e}^{-\mathrm{i}S_0(0,t)} \tag{59}$$

where we can use equation (40) to obtain

$$\tilde{\Phi}_m^{({\rm i})'} = C_m\frac{(r-r_m) + \Delta_m^{({\rm i})}/\alpha_m}{(r-r_m)^2 + (\Delta_m^{({\rm i})})^2}. \tag{60}$$

Since $\Delta_{\rm b}^{\rm p} \ll |r_m/m|$, we neglect the drift frequency in the resonance condition for well-passing particles and approximate $S_0(0,t) \simeq s(\pi+\theta)$. Notice that the dominant contribution to the growth rate comes from the resonances $s = 1$ and $s = -1$. Changing variables in equation (59) according to $r = r_0 + \Delta_{\rm b}^{\rm p}\cos\theta$ and $\theta = \omega_{\rm b} t$, we arrive at

$$G_m = (-1)^s\frac{v_{\rm D}}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\,\tilde{\Phi}_m^{(i)'}(r_0 + \Delta_{\rm b}^{\rm p}\cos\theta)\sin\theta\, \mathrm{e}^{-\mathrm{i}s\theta}. \tag{61}$$

Then it can be shown to the lowest order in ϵ_m that

$$\int_{J_\varphi 1}^{J_\varphi 2}\mathrm{d}J_\varphi\,|G_m|^2\left(\frac{\omega_{\rm r}}{\omega_\star^{\rm p}} - 1\right)\frac{\partial f_0}{\partial J_\varphi} \simeq \frac{C_m^2}{\Delta_m^{({\rm i})}}\left[\frac{v_{\rm D}^2\epsilon_m}{\pi^2 q}I(D^{\rm p})\left(\frac{\omega_{\rm r}}{\omega_\star^{\rm p}} - 1\right)\frac{\partial f_0}{\partial J_\varphi}\right]\bigg|_{J_\varphi=\Psi_m} \tag{62}$$

where $J_{\varphi 1,2} = \psi_m - (v/\omega_c)[1 - \lambda \mp (2-\lambda)\epsilon_m]^{1/2}$ with $\psi_m \simeq r_m^2/qR_0$ and $\lambda = \mu B_0/E$

$$I(D) = \int_{-D}^{D} \mathrm{d}y \left| \int_0^{\pi} \mathrm{d}\theta \, \frac{(y + D\cos\theta) + 1/\alpha_m}{(y + D\cos\theta)^2 + 1} \sin^2\theta \right|^2 \tag{63}$$

and $D^{\mathrm{p}} = \Delta_{\mathrm{b}}^{\mathrm{p}}/\Delta_m^{(\mathrm{i})}$. The function $I(D)$ is evaluated numerically for $\alpha_m = 3$ and shown in figure 1.

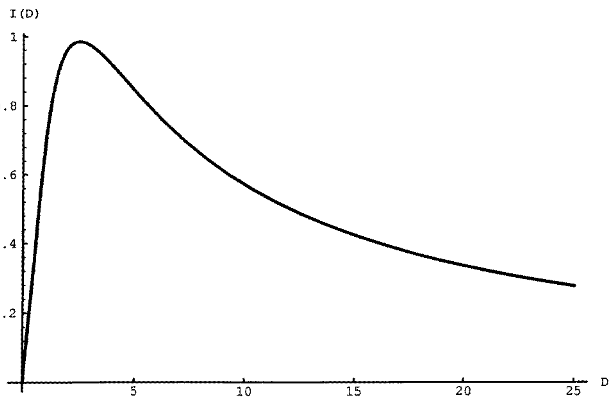

Figure 1. Numerically calculated integral $I(D)$ for $\alpha_m = 3$.

Substituting (62) into (38) together with

$$\int_{r_m-\Delta_{\mathrm{b}}}^{r_m+\Delta_{\mathrm{b}}} \mathrm{d}r\, r |\tilde{\Phi}_m^{(\mathrm{i})'}|^2 \simeq C_m^2 \frac{\pi r_m}{2\Delta_m^{(\mathrm{i})}} \tag{64}$$

and taking into account the fact that $I(D^{\mathrm{p}}) \sim 2\pi^2/3D^{\mathrm{p}}$ for $D^{\mathrm{p}} \gg 1$ we obtain

$$\frac{\gamma_{\mathrm{p}}^n}{\omega} = \frac{128\pi^2 q^3 M n}{3B_0^2\omega_{\mathrm{c}}} \sum_{v_{\mathrm{r}}=v_{\mathrm{A}}, v_{\mathrm{A}}/3} \int_0^{v_{\mathrm{r}}/\sqrt{2\epsilon}} \mathrm{d}v_\perp \frac{\Delta_m^{(\mathrm{i})}}{\Delta_{\mathrm{b}}^{\mathrm{p}}} J_0^2\left(\frac{m v_\perp}{r_m \omega_{\mathrm{c}}}\right) \times \frac{v_\perp v_{\mathrm{r}}}{v_{\mathrm{A}}} \left(v_{\mathrm{r}}^2 + \frac{v_\perp^2}{2}\right)^2 \left(\frac{\omega_{\mathrm{r}}}{\omega_\star^{\mathrm{p}}} - 1\right) \frac{\partial f_0}{\partial J_\varphi}\bigg|_{\substack{v_\parallel = v_1 \\ J_\varphi = \psi_m}}. \tag{65}$$

Comparing the above expression with the corresponding result of the local theory, equation (50), one finds that the growth rate is reduced by a factor $D = \Delta_{\mathrm{b}}^{\mathrm{p}}/\Delta_m^{(\mathrm{i})}$ when the excursion of particle orbits $\Delta_{\mathrm{b}}^{\mathrm{p}}$ significantly exceeds the width of the inner mode localization $\Delta_m^{(\mathrm{i})}$. A similar result has been obtained in [13] where the instability drive has been calculated by using a Fourier decomposition of the inner mode structure.

4.2.2. Trapped particles. Similarly to the case of passing particles we approximate the function G_m by

$$G_m \simeq \int_0^{\tau_{\mathrm{b}}^{\mathrm{t}}} \frac{\mathrm{d}t}{\tau_{\mathrm{b}}^{\mathrm{t}}} \tilde{\Phi}_m^{(\mathrm{i})'}(r) v_{\mathrm{D}} \sin\theta \, \mathrm{e}^{-\mathrm{i}S_0(0,t)}. \tag{66}$$

Using the resonance condition (52) we find for well-trapped particles ($\kappa^2 \ll 1$)

$$\mathrm{e}^{-\mathrm{i}S_0(0,t)} \simeq 2\cos\left[s\left(\frac{\pi}{2} + \alpha\right)\right] \exp \mathrm{i}[\arcsin\kappa + \arcsin(\kappa \sin\alpha)] \tag{67}$$

where $\alpha = \arcsin((1/\kappa)\sin(\theta/2))$ and $\omega_b^t t \simeq (\pi/2) + \alpha$. Considering the $s = 1$ resonance, we obtain from equations (60), (66) and (67) to the lowest order in ϵ_m

$$\int_{J_\varphi 1}^{J_\varphi 2} \mathrm{d}J_\varphi |G_m|^2 \left(\frac{\omega_r}{\omega_\star^t} - 1\right)\frac{\partial f_0}{\partial J_\varphi} \simeq \frac{C_m^2}{\Delta_m^{(i)}}\left[\frac{4v_D^2\kappa^2\epsilon_m}{\pi^2 q} I(D^t)\left(\frac{\omega_r}{\omega_\star^p} - 1\right)\frac{\partial f_0}{\partial J_\varphi}\right]\Bigg|_{J_\varphi = \psi_m} \tag{68}$$

where $I(D)$ is given by equation (63), $D^t = \Delta_b^t/2\Delta_m^{(i)}$, $J_{\varphi 1,2} = \psi_m \pm (v/\omega_c)\kappa\sqrt{2\epsilon_m}$, $v_D \simeq v^2/2\omega_c R_0$ and $\Delta_b^t = 2(2/\epsilon_m)^{1/2} q_m \kappa v/\omega_c$ is the banana width at a given value of κ. Substituting (68) and (64) into (38) and using the approximation $I(D^t) \simeq 2\pi^2/(3D^t) = 4\pi^2\Delta_m^{(i)}/(3\Delta_b^t)$, for $D^t \gg 1$, we get the following 'nonlocal' expression for the TAE growth rate due to trapped energetic ions:

$$\frac{\gamma_t^n}{\omega} = \frac{256\pi^2\sqrt{2\epsilon}q^2 n M v_1^6}{3B_0^2 R_0 \omega_c D_1} J_0^2\left(\frac{m v_1}{r\omega_c}\right)\int_0^{\kappa_{\max}^2} \kappa^2 \mathrm{d}\kappa^2 \frac{\Delta_m^{(i)}}{\Delta_b^t}\left(\frac{\omega_r}{\omega_\star^t} - 1\right)\frac{\partial f_0}{\partial J_\varphi}\Bigg|_{\substack{v = v_1 \\ J_\varphi = \psi_m}} \tag{69}$$

where v_1 and D_1 are defined by equations (56) and (58), respectively, and all quantities are calculated at $r = r_m$. From the expressions (57) and (69) one can estimate that in the considered limit the nonlocal theory predicts a reduction of the growth rate by a factor $\Delta_m^{(i)}/(\Delta_b^t)_{\max}$ as compared to the result of the local theory, where $(\Delta_b^t)_{\max} \simeq 2\sqrt{2} q_m v \kappa_{\max}/(\omega_c \epsilon_m^{1/2})$ is the maximum banana width.

4.3. Nonlocal theory: $\Delta_m^{(o)} \leqslant \Delta_b < r_m$

4.3.1. Passing particles. As long as the particle orbit excursion from the magnetic surface Δ_b^p exceeds the outer mode width $\Delta_m^{(o)}$, the function G_m is mainly determined by the outer mode structure. Making use of the resonance condition (46) and changing the variables according to $r = r_0 + \Delta_b \cos\theta$ and $\theta = \omega_b^p t$ we obtain from equation (45)

$$G_m = -\mathrm{i}\frac{s\omega_b^p}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \tilde{\Phi}_m^{(o)}(r_0 + \Delta_b^p\cos\theta)\mathrm{e}^{-\mathrm{i}s\theta}\mathrm{e}^{\mathrm{i}(\Delta_b^p m/r_0)\sin\theta} \tag{70}$$

with $\tilde{\Phi}_m^{(o)}$ given by equation (41). The denominator of equation (38) is proportional to the mode energy which is mostly determined by the inner mode structure. We then substitute equations (70) and (64) into equation (38) and obtain

$$\frac{\gamma_p^{nl}}{\omega} = \frac{64q^3 M n}{B_0^2\omega_c}\sum_{v_r = v_A, v_A/3}\int_0^{v_r/\sqrt{2\epsilon}} \mathrm{d}v_\perp \frac{\Delta_m^{(i)}}{\Delta_b^p} J_0^2\left(\frac{m v_\perp}{r_m\omega_c}\right)\frac{v_\perp v_r}{v_A}\left(v_r^2 + \frac{v_\perp^2}{2}\right)^2 \frac{H(W^p)}{W^p}$$
$$\times \left(\frac{\omega_r}{\omega_\star^p} - 1\right)\frac{\partial f_0}{\partial J_\varphi}\Bigg|_{\substack{v_\parallel = v_1 \\ J_\varphi = \psi_m}} \tag{71}$$

where the coupling integral $H(W^p)$ is defined by

$$H(W^p) = \int_{-W^p}^{W^p} \mathrm{d}y\left|\int_0^\pi \mathrm{d}\theta\, K_0(|y + W^p\cos\theta|)\cos(\theta + W^p\sin\theta)\right|^2 \tag{72}$$

with $W^p = \Delta_b^p/\Delta_m^{(o)}$. The function $H(W^p)$ is shown in figure 2. A similar result has been obtained in [17], where the function $H(W^p)$ has also been calculated analytically for large values of W^p

$$H(W^p) \simeq \frac{\pi^2}{W^p} \qquad W^p \gg 1. \tag{73}$$

Consequently it follows from equation (71) and (72) that for $W^p \gg 1$ the growth rate reduction is of the order of $\Delta_m^{(i)}/\Delta_b^p (W^p)^2$ as compared to the result of the local theory, equation (50). On the other hand, the result of the nonlocal theory in the limit $\Delta_m^{(i)} \ll \Delta_b^p \ll \Delta_m^{(o)}$, equation (65), follows directly from equation (71) for

$$H(W^p) \simeq \frac{2\pi^2}{3} W^p \qquad W^p \ll 1. \tag{74}$$

4.3.2. Trapped particles. Similarly to the case of passing particles we neglect here the finite FLR effect and assume that the function G_m is determined by the outer mode structure. Then using equations (64) and (67) we obtain from equation (38) for the $s = 1$ resonance

$$\frac{\gamma_t^{nl}}{\omega} = \frac{128\sqrt{2\epsilon}q^2 n M v_1^6}{B_0^2 R_0 \omega_c D_1} J_0^2\left(\frac{m v_1}{r\omega_c}\right) \int_0^{\kappa_{max}^2} \kappa^2\,d\kappa^2\, \frac{\Delta_m^{(i)}}{\Delta_b^t} \frac{H(W^t)}{W^t} \left(\frac{\omega_r}{\omega_\star^t} - 1\right) \frac{\partial f_0}{\partial J_\varphi}\bigg|_{\substack{v=v_1\\ J_\varphi=\Psi_m}} \tag{75}$$

where $W^t = \Delta_b^t/2\Delta_m^{(o)}$, v_1 and D_1 are given by equations (56) and (58), respectively, and all quantities are calculated at $r = r_m$. The function $H(W^t)$ being defined as

$$H(W^t) = \frac{1}{4}\int_{-W^t}^{W^t} dy \left| \int_{-\pi/2}^{\pi/2} d\alpha\,[K_0(|y + W^t\cos\alpha|)e^{-i(\alpha+\pi/2)} + K_0(|y - W^t\cos\alpha|)e^{i(\alpha+\pi/2)}] \times e^{i\kappa\sin\alpha}\right|^2 \tag{76}$$

is evaluated numerically for $\kappa = 0$ and shown in figure 3. For small values of W^t, this integral can be evaluated analytically by approximating $K_0(y) \simeq \pi\delta(y)$ to obtain

$$H(W^t) \simeq \frac{2\pi^2}{W^t} \qquad W^t \gg 1. \tag{77}$$

For small values of W^t we find

$$H(W^t) \simeq \frac{2\pi^2}{3} W^t \qquad W^t \ll 1. \tag{78}$$

Using this expression in equation (75) we recover the previously obtained result in the limit $\Delta_m^{(i)} \ll \Delta_b^t \ll \Delta_m^{(o)}$ as given by equation (69).

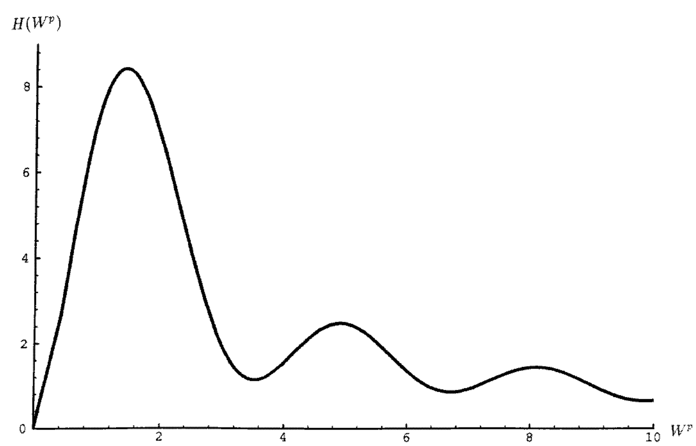

Figure 2. Numerically calculated integral $H(W^p)$ for passing particles.

5. TAE excitation by energetic particles with a slowing-down distribution function

In this section, we apply the derived growth rate expressions to the case of energetic ions with a slowing-down distribution function generated either by NBI or by an isotropic source such as fusion produced alpha particles. In order to clearly expose the stabilization effect due to large particle orbit widths, we will only consider here the driving part of the growth rate and perform the calculations for the TFTR parameters, cf [22]: $R_0 = 240$ cm, $a = 75$ cm, $B_0 = 1$ T, $n_e(0) = 2.7\times10^{13}$ cm^{-3}, $m_{\rm eff} = 2$, $Z_{\rm eff} = 2.5$. We also assume that a $m = 3$, $n = 2$ TAE mode is excited at $r_m = 30$ cm and $q_m = 1.25$. The corresponding widths of the inner and outer mode structure are then $\Delta_m^{(\rm i)} \simeq 4.4$ cm (with $d = 3.5$) and $\Delta_m^{(\rm o)} \simeq 10$ cm, respectively.

5.1. Strongly anisotropic slowing-down distribution of fast ions

For NBI-generated ions we take the velocity distribution function in the form

$$f_0 = \frac{3\beta_{\rm b}B_0^2}{8\pi^2 M v_0^2}\frac{\eta(v_0 - v)}{v^3}\delta(\chi - \chi_0) \tag{79}$$

where $\chi = v_\parallel/v$, $\eta(x)$ is the step function, v_0 is the injection velocity and χ_0 is the injection pitch angle. The distribution function f_0 is normalized so that $\beta_{\rm b} = (8\pi M/3B_0^2)\int v^2 f_0\,{\rm d}^3v$ is the mean beta of the beam particles. Since for passing particles $\xi = mv_\perp/r_m\omega_{\rm c} \ll 1$, we can neglect the small stabilizing FLR effect by approximating $J_0^2(\xi) \simeq 1$. Substituting (79) into the driving part of the previously derived growth rate expressions (50), (65) and (71) we obtain for the excitation by passing ions

$$\left(\frac{\gamma_{\rm p}}{\omega_{\rm r}}\right)_{\rm drive} = -q_m^2\beta_{\rm b}^{\rm p}\frac{\omega_{\star b}^{\rm p}}{\omega_{\rm r}}(1+\chi_0^2)^2$$

$$\times\begin{cases} \dfrac{3\pi}{4}\displaystyle\sum_{s=\pm1} y_s^4\eta(1-y_s) & \Delta_{\rm b}^{\rm p} \ll \Delta_m^{(\rm i)} \qquad (80a)\\ 4\dfrac{\Delta_m^{(\rm i)}}{\zeta_{\rm p}}\displaystyle\sum_{s=\pm1} y_s^3\eta(1-y_s) & \Delta_m^{(\rm i)} \ll \Delta_{\rm b}^{\rm p} \ll \Delta_m^{(\rm o)} \qquad (80b)\\ 24\dfrac{\Delta_m^{(\rm i)}(\Delta_m^{(\rm o)})^2}{\zeta_{\rm p}^3}\displaystyle\sum_{s=\pm1} y_s\eta(1-y_s) & \Delta_{\rm b}^{\rm p} \gg \Delta_m^{(\rm o)} \qquad (80c)\end{cases}$$

where $y_1 = v_{\rm A}/(\chi_0 v_0)$, $y_{-1} = v_{\rm A}/(3\chi_0 v_0)$, $\zeta_{\rm p} = (1+\chi_0^2)qv_0/(\chi_0\omega_{\rm c})$ and $\omega_{\star b}^{\rm p} = (nq_m v_0^2/r_m\omega_{\rm c})\partial\ln\beta_{\rm b}^{\rm p}/\partial r$, with the superscript ‘p’ denoting passing particles. For the tangential NBI injection with $\chi_0 = 1$ and $v_0 = 3.2\times10^8$ cm s^{-1} ($E_0 = 110$ keV), we have $\Delta_{\rm b}^{\rm p} \simeq 8.3$ cm and thus $\Delta_m^{(\rm i)} < \Delta_{\rm b}^{\rm p} < \Delta_m^{(\rm o)}$. Since for the considered parameters $v_{\rm A} = 3\times10^8$ cm s^{-1}, the instability excitation condition $y_s < 1$ is satisfied for both $|v_\parallel| = v_{\rm A}$ and $|v_\parallel| = v_{\rm A}/3$ resonances. However, the contribution from the $|v_\parallel| = v_{\rm A}/3$ resonance in equations (80*a*) and (80*b*) is negligibly small. Comparing the growth rate values calculated from equations (80*a*) and (80*b*), we find that $\gamma_{\rm p}^{\rm l}/\gamma_{\rm p}^{\rm nl} \sim 1$, i.e. the particle orbit width effect is not sufficiently strong for the considered parameters to provide stabilization. This conclusion is in agreement with the numerical results of [14].

In the case of TAE excitation by trapped NBI ions, equations (57), (69) and (75) together with (79) yield

$$\left(\frac{\gamma_{\rm t}}{\omega_{\rm r}}\right)_{\rm drive} = -q_m\beta_{\rm b}^{\rm t}\frac{v_{\rm A}}{R_0}\frac{\omega_{\star\rm b}^{\rm t}}{\omega_{\rm r}}\frac{v_1^3}{D_1v_0^4}J_0^2\left(\frac{mv_1}{r_m\omega_{\rm c}}\right)\eta(v_0 - v_1)$$

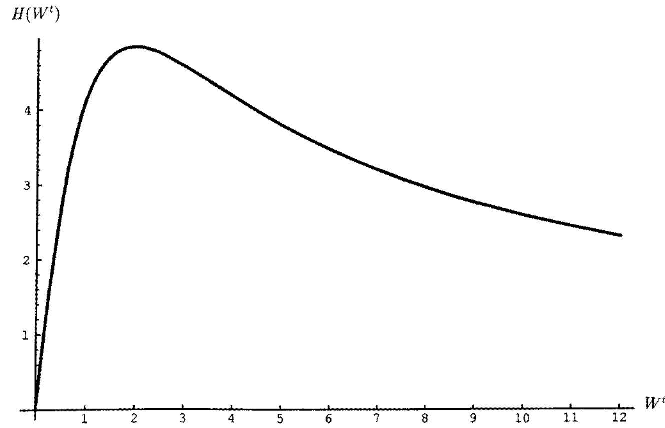

Figure 3. Numerically calculated integral $H(W^{\rm t})$ for trapped particles with $\kappa = 0$.

$$\times \begin{cases} \dfrac{3\pi}{2\sqrt{2}} & (\Delta_{\rm b}^{\rm t})_{\rm max} \ll \Delta_m^{\rm (i)} \quad (81a) \\ 16\dfrac{\Delta_m^{\rm (i)}}{\zeta_{\rm t}} & \Delta_m^{\rm (i)} \ll (\Delta_{\rm b}^{\rm t})_{\rm max} \ll \Delta_m^{\rm (o)} \quad (81b) \\ 384\dfrac{\Delta_m^{\rm (i)}(\Delta_m^{\rm (o)})^2}{\zeta_{\rm t}^3} & (\Delta_{\rm b}^{\rm t})_{\rm max} \gg \Delta_m^{\rm (o)} \quad (81c) \end{cases}$$

where $\omega_{\star{\rm b}}^{\rm t} = (nq_m v_0^2/r_m\omega_{\rm c})\partial \ln \beta_{\rm b}^{\rm t}/\partial r$, $\zeta_{\rm t} = 2(2/\epsilon_m)^{1/2} q_m v_1/\omega_{\rm c}$, and we have used $\chi_0 \simeq \sqrt{\epsilon_m}$. The expressions for v_1 and D_1 are given by equations (56) and (58), respectively. For the considered plasma parameters we obtain $v_1 = 2.74 \times 10^8$ cm s^{-1}, which shows that the excitation condition $v_0 > v_1$ can only be satisfied if the effect of particle precessional drift motion is taken into account in the resonance condition. Since $\zeta_{\rm t} = 57$ cm and thus $(\Delta_{\rm b}^{\rm t})_{\rm max} > \Delta_m^{\rm (o)}$, we find from equations (81a) and (81c) that $\gamma_{\rm t}^{\rm l}/\gamma_{\rm t}^{\rm nl} \sim 4$, which predicts a stabilizing effect which is about two times weaker than that obtained in the numerical simulations of [14]. The FLR effect for trapped particles gives a reduction of the growth rate of about 25%.

5.2. Isotropic slowing-down distribution of fast ions

For fusion-generated alpha particles we consider the following velocity distribution function

$$f_0 = \frac{S_\alpha \tau_{\rm s}}{4\pi} \frac{\eta(v_\alpha - v)}{v^3} \quad (82)$$

where $S_\alpha = n^2\langle\sigma v\rangle/4$, $\langle\sigma v\rangle$ is the fusion reactivity, $\tau_{\rm s}$ is the alpha particle slowing-down time, $n_\alpha = S_\alpha\tau_{\rm s}$ is the alpha particle density and $v_\alpha = 1.3 \times 10^9$ cm s^{-1} is the alpha particle birth velocity.

In the case of TAE excitation by passing alpha particles, we obtain from equation (50) in the limit $\Delta_{\rm b}^{\rm p} \ll \Delta_m^{\rm (i)}$

$$\left(\frac{\gamma_{\rm p}}{\omega_{\rm r}}\right)_{\rm drive} = -\frac{\pi}{16} q_m^2 \beta_\alpha^{\rm p} \frac{\omega_{\star\alpha}^{\rm p}}{\omega_{\rm r}}$$

$$\times \sum_{s=\pm1} \begin{cases} \dfrac{y_s^4}{(2\epsilon_m)^{3/2}}(1 + 12\epsilon_m - 4(2\epsilon_m)^{3/2}) & y_s < \sqrt{2\epsilon_m} \quad (83a) \\ y_s(1 + 6y_s^2 - 4y_s^3 - 3y_s^4)\eta(1 - y_s) & y_s > \sqrt{2\epsilon_m} \quad (83b) \end{cases}$$

where $y_1 = v_A/v_\alpha$, $y_{-1} = v_A/3v_\alpha$, $\beta_\alpha^p = 4\pi n_\alpha v_\alpha^2 M_\alpha/3B_0^2$ and $\omega_{\star\alpha}^p = (nq_m v_\alpha^2/r_m\omega_c)\ln\beta_\alpha^p\partial/\partial r$. Note that the FLR effect has been neglected in the above expression by approximating $J_0^2(\xi) \simeq 1$. Then using (65) and (71), it follows that the above result should be multiplied by approximately a factor of $(16/3\pi)(\Delta_m^{(i)}/q_m\rho_\alpha)$ when $\Delta_m^{(i)} \ll \Delta_b^p \ll \Delta_m^{(o)}$, and by a factor $(8/\pi)\Delta_m^{(i)}(\Delta_m^{(o)})^2/(q_m\rho_\alpha)^3$ when $\Delta_b^p \gg \Delta_m^{(o)}$, where $\rho_\alpha = v_\alpha/\omega_c$. Since, for the TFTR parameters mentioned previously $\Delta_b^p \gg \Delta_m^{(o)}$, it follows that the local instability drive is reduced by a factor of the order of 25 due to the finite orbit width effect.

In the case of TAE excitation by trapped alpha particles, equations (57), (69) and (75) together with (82) predict

$$\left(\frac{\gamma_t}{\omega_r}\right)_{\rm drive} = -\frac{3\pi}{8}(2\epsilon_m)^{1/2}\frac{\omega_{\star\alpha}}{\omega_r}\frac{q_m\beta_\alpha^t v_A v_1^3}{R_0 D_1 v_\alpha^4}\kappa_{\max}^4\eta(v_\alpha - v_1)J_0^2\left(\frac{mv_1}{r_m\omega_c}\right)$$

$$\times\begin{cases} 1 & (\Delta_b^t)_{\max} \ll \Delta_m^{(i)} \qquad (84a) \\ \dfrac{64}{9\pi}\dfrac{\Delta_m^{(i)}}{\zeta_t} & \Delta_m^{(i)} \ll (\Delta_b^t)_{\max} \ll \Delta_m^{(o)} \qquad (84b) \\ 256\dfrac{\Delta_m^{(i)}(\Delta_m^{(o)})^2}{\zeta_t^3} & (\Delta_b^t)_{\max} \gg \Delta_m^{(o)} \qquad (84c) \end{cases}$$

where $\zeta_t = 2(2/\epsilon_m)^{1/2}q_m v_1/\omega_c$, v_1 and D_1 are defined by (56) and (58), respectively. For the considered parameters, the excitation condition $v_\alpha > v_1$, is satisfied even if the effect of particle precessional drift motion is neglected in the expressions for v_1 and D_1. Then, we obtain for the TFTR parameters that $v_1 = v_A/\sqrt{2\epsilon_m} = 6 \times 10^8$ cm s^{-1} and $\zeta_t = 15.6$ cm which in the appropriate limit, $(\Delta_b^t)_{\max} \gg \Delta_m^{(o)}$, give a reduction of the local instability drive by a factor of about nine.

6. Conclusions

We have presented a generalized method to predict the linear instability growth rate of the TAE modes in large-aspect-ratio and low-beta plasma tokamak equilibrium. The approach is based on employing the Hamiltonian action-angle variables and takes into account the finite orbit width effects, the FLR effects of energetic particles and the effect of mode localization, as well as the possible mode excitation by both passing and trapped energetic ions.

Particular attention has been devoted to the stabilizing effect of large particle orbit widths on the TAE instability drive. Two limiting cases, when $\Delta_m^{(i)} \ll \Delta_b \ll \Delta_m^{(o)}$ and $\Delta_b \gg \Delta_m^{(o)}$, have been analysed and compared with the prediction of the narrow orbit theory, when $\Delta_b \ll \Delta_m^{(i)}$, where Δ_b is the particle orbit width and $\Delta_m^{(i)}$ and $\Delta_m^{(o)}$ are the inner and outer widths of the mode structure, respectively. The corresponding reduction of the instability drive in these cases is of the order of $\Delta_m^{(i)}/\Delta_b$ and $\Delta_m^{(i)}(\Delta_m^{(o)})^2/\Delta_b^3$ for excitation by both passing and trapped energetic particles. This conclusion is in agreement with the existing analytical theory for excitation by passing fast ions [17, 32] as well as with numerical simulations [14, 16].

An application of our results to a slowing-down distribution of energetic ions generated by NBI and of fusion-produced alpha particles for the TFTR parameters shows that the stabilization of the TAE instability due to large particle orbit width is much more pronounced in the case of an isotropic alpha particle distribution. This fact may be of particular importance for interpretation of the TFTR experiments, where the instability has been observed during NBI heating, whereas the TAE excitation by fusion-produced alpha particles could not be demonstrated. However, to make a reasonable prediction about the instability thresholds, it is necessary to include in the theory contributions from various damping mechanisms, a problem which is beyond the scope of the present work.

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