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Slowing-Down Dynamics of Fast Particles in Plasmas via the Fokker-Planck Equation

D. Anderson, M. Lisak, F. Andersson,* and T. Fülöp

Chalmers University of Technology, Department of Electromagnetics and EURATOM-VR Association, SE-412 96 Göteborg, Sweden

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Abstract—*A detailed discussion is given of the effects of energy diffusion and pitch-angle scattering on the slowing-down dynamics of a beam of monoenergetic particles being released with unidirectional velocity. Approximate solutions are given for characteristic averaged quantities like the pitch-angle averaged distribution function and different physically relevant velocity moments. The relation to previous exact investigations is discussed.*

I. INTRODUCTION

The collisional slowing down of a beam of high-energy charged particles in a background plasma is a problem of fundamental importance with applications ranging from fusion plasmas to astrophysical plasmas. For example, in fusion plasmas, examples of such high-energy beams are abundant: fusion-generated alpha particles, neutral-beam-injected particles, ion cyclotron resonance frequency–heated particles, and runaway electrons. In two papers, Manservisi and Molinari[1,2] consider the slowing-down dynamics of a group of fast particles being released with a unidirectional velocity. The analysis is based on the Fokker-Planck equation, which describes the collisional dynamics of the particles, including the effects of frictional slowing down, energy diffusion, and pitch-angle scattering. It is well known that certain timescales can be introduced to characterize these physical effects, but these concepts are not always clearly defined ones (see Refs. 3 and 4). The analysis in Refs. 1 and 2 considers the slowing-down dynamics of the beam in more detail using an analytical solution of the Fokker-Planck equation. This solution is then used to evaluate slowing-down characteristics given as certain physically relevant velocity moments of the distribution function and to derive some of the aforementioned characteristic timescales. A drawback of this analysis is the fact that the analytical solution is very cumbersome and does not convey a clear picture of the physical processes involved in the slowing-down dynamics. The purpose of the present work is to reconsider this problem, using an approach based on approximate analytical solutions, which give simple analytical expressions for the characteristic physical quantities and furthermore give a clear picture of the physics of the problem.

*E-mail: fredrik.andersson@elmagn.chalmers.se

II. THE FOKKER-PLANCK EQUATION

The slowing-down dynamics of a beam of particles being released at time $t = 0$ with a unidirectional velocity $v = v_0$ can be described by the following Fokker-Planck equation for the normalized test particle distribution $f = f(v, \mu, t)$, where $v = |\mathbf{v}|$ and μ is the cosine of the pitch angle relative to the initial direction of the particle beam velocity:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left[\frac{\alpha}{v^2} f + \frac{\partial}{\partial v}\left(\frac{\beta}{v} f\right)\right] + \frac{\gamma}{v^3}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial f}{\partial \mu}\right] + S \ . \quad (1)$$

In this form of the Fokker-Planck equation, which was used in Refs. 1 and 2, the distribution function, $f(v, \mu, t)$, is defined in such a way that the volume element in velocity space is $dvd\mu$, rather than $v^2 dvd\mu$. The source term S is given by

$$S = N\delta(v - v_0)\delta(\mu - 1)\delta(t) \ ,$$

where N is the total number of test particles released.

In the considered situation, the particle beam is assumed to have small density and particle velocities that are much larger than those of the background particles, which are taken to be zero. In this limit, the coefficients α, β, and γ become independent of velocity and are given by Refs. 1 and 2:

$$\alpha = nM\theta[1 - M(1 - 1/(2\ln\Lambda))] \ ,$$

$$\beta = nM^2\theta/(2\ln\Lambda) \ ,$$

and

$$\gamma = nM^2\theta[1 - 1/(2\ln\Lambda)]/2 \ ,$$

where $\ln\Lambda$ is the Coulomb logarithm, n is the background particle density, and

$$\theta = \frac{e^4 Z^2 \ln\Lambda}{4\pi\epsilon^2 m'^2}$$

and

$$M = \frac{m'}{m_t} = \frac{m_b}{m_b + m_t} \ ,$$

where m' is the reduced mass

$$\frac{1}{m'} = \frac{1}{m_b} + \frac{1}{m_t} \ ,$$

and the indices b and t refer to background and test particles, respectively.

The relative magnitude of the terms corresponding to friction, energy diffusion, and pitch-angle scattering, respectively, can be estimated by comparing the coefficients α, β, and γ as

$$\alpha : \beta : \gamma = 1 : \frac{M}{2(1 - M)\ln\Lambda} : \frac{M}{2(1 - M)} \ ,$$

where it is assumed that $\ln\Lambda \gg 1$. Depending on the physical situation, different physical effects may dominate. Consider, for example, the case of heavy particles slowing down on light particles, e.g., fusion-generated alpha particles slowing down on electrons. In this case, $M \ll 1$ and the friction term of the Fokker-Planck equation describes the most important effect for the dynamics. On the other hand, for light particles slowing down on heavy particles, e.g., electrons slowing down on ions, then clearly pitch-angle scattering is the dominant effect. The solution of the Fokker-Planck equation in the general case can be obtained using separation of variables (see Refs. 1 and 2). However, the corresponding expression becomes a double sum over terms involving the eigenfunctions and the corresponding eigenvalues of the separated equations. Albeit exact, this solution is exceedingly complicated and does not convey a clear picture of the physics involved in the dynamics. Nevertheless, in Refs. 1 and 2, this solution forms the basis for an investigation of different general properties of the beam dynamics, in particular, the characteristic timescales for slowing down t_s, energy diffusion t_d, and pitch-angle scattering t_p.

In the present analysis we reconsider this problem using a different and approximate approach, which gives a clear physical picture of the slowing down of the beam.

A good way of examining and characterizing the slowing-down process is to calculate the evolution of certain velocity space moments, e.g., the average energy of the beam distribution. This approach is also pursued in Refs. 1 and 2. In fact, they define one of the timescales, the slowing down time t_s, as the time taken for the energy of the beam distribution $E = E(t) = \langle mv^2/2 \rangle$ to reach the thermal energy E_{th} of the background particles. Here, the averaged quantity or velocity space moment $\langle G(v, \mu) \rangle$ is defined by

$$\langle G(v,\mu) \rangle = \frac{\displaystyle\int G(v,\mu) f(v,\mu,t)\, dv\, d\mu}{\displaystyle\int f(v,\mu,t)\, dv\, d\mu} \ .$$

An analysis based on velocity space moments is presented in Secs. IV and VI, but already at this point a first good estimate of the expected timescales t_s, t_d, and t_p can easily be obtained by writing the Fokker-Planck Eq. (1) in the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left[\frac{\alpha - \beta}{v^2} f + \frac{\beta}{v}\frac{\partial f}{\partial v}\right] + \frac{\gamma}{v^3}\frac{\partial}{\partial \mu}\left[(1 - \mu^2)\frac{\partial f}{\partial \mu}\right] + S \ . \qquad (2)$$

From this we can directly infer the following qualitative estimates of the characteristic timescales for the different dynamical processes involved in the slowing down of the beam, namely,

$$t_s \approx \frac{v_0^3}{(\alpha - \beta)} \ ,$$

$$t_d \approx \frac{v_0^3}{\beta} \ ,$$

and

$$t_p \approx \frac{v_0^3}{\gamma} \ . \qquad (3)$$

It is instructive to consider the Fokker-Planck Eq. (1) in the case when the friction force dominates the particle dynamics. We will find that although the slowing-down time definition given in Refs. 1 and 2 is consistent in this limit, the situation becomes less clear when energy diffusion is taken into account.

III. THE BENCHMARKING CASE OF COLLISIONAL SLOWING DOWN ONLY

In cases where the friction force is the dominant term in determining the slowing-down behavior, the Fokker-Planck Eq. (1) for the full particle distribution $f = f(v, \mu, t)$ can be approximated as

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left(\frac{\alpha}{v^2} f\right) + N\delta(v - v_0)\delta(\mu - 1)\delta(t) \ .$$

This equation is easily solved using the characteristic coordinate $\tau = t + v^3/(3\alpha)$, and the solution is

$$f(v, \mu, t) = N\delta(v - v_f(t))\delta(\mu - 1) \ ,$$

where

$$v_f(t) = v_0\left(1 - \frac{3\alpha t}{v_0^3}\right)^{1/3} \ . \tag{4}$$

Clearly, this solution describes a particle velocity distribution where the velocity dependence is in the form of a delta function slowing down continuously with a velocity $v_f(t)$. If the slowing-down time t_s is defined by the condition $v_f(t) = 0$, one obtains the classical result: $t_s \equiv v_0^3/(3\alpha)$, in qualitative agreement with the dimensional estimate given in Eq. (3) for $\beta = 0$. During slowing down, the beam does not spread, neither in energy nor in pitch angle, i.e., the parallel and perpendicular temperatures of the beam remain zero.

The different velocity moments corresponding to the weighting functions v^n become

$$\langle v^n \rangle = v_0^n\left(1 - \frac{t}{t_s}\right)^{n/3} \ .$$

In particular, we note that the third-order moment has an especially simple form, decaying linearly to zero according to

$$\langle v^3 \rangle = v_0^3\left(1 - \frac{t}{t_s}\right) \ .$$

At this stage, the energy definition of the slowing-down time seems noncontroversial insofar as all velocity moments vanish at time $t = t_s$. However, at closer inspection we infer that there is a significant difference between the time behavior of moments of different order. The lowest-order moment ($n = 0$), corresponding to the particle density, remains constant until $t = t_s$, when all fast particles instantaneously reach zero velocity, i.e., become thermalized. On the other hand, the higher-order moments decay continuously with time, and the rate of decay increases with increasing order (see Fig. 1). In fact, the initial decay of the moments proceeds according to

$$\langle v^n \rangle \approx v_0^n\left(1 - \frac{n}{3}\frac{t}{t_s}\right) \ ,$$

i.e., the characteristic timescale for the initial variation is $t_s^{(n)} = 3t_s/n$.

With this picture in mind, it is clear that there has been a more-or-less significant change of the moments (depending on the order) even before they eventually vanish at time $t = t_s$. However, except for the higher-order moments, the timescale for significant changes is still of the order of the classical slowing down time t_s.

The situation becomes more complicated when diffusion effects are taken into account. Intuitively, we expect diffusion to smooth out the discrete vanishing of the moments at $t = t_s$ and that $\langle v^n \rangle(t) \to 0$ only as $t \to \infty$. This intuitive picture is indeed confirmed by the solution derived in Sec. V for the second-order velocity moment in the presence of diffusion effects (see Fig. 2). Clearly, there is now no longer a finite time at which any of the moments vanish.

In Refs. 1 and 2, the concept of slowing-down time in this situation is taken as the time $\hat{t}_s$ at which the second-order energy moment is equal to the thermal energy of the background particles, which implies that

$$\langle v^2 \rangle(\hat{t}_s) = v_{th}^2 \ ,$$

where v_{th} is the thermal velocity of the background particles. However, the Fokker-Planck coefficients used in the analysis correspond to those of a cold background, i.e., $v_{th} = 0$. Furthermore, since v_{th} is assumed to be close

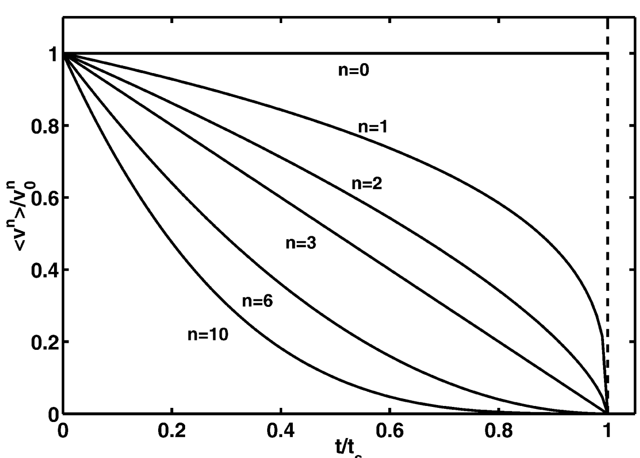

Fig. 1. The time variation of different velocity moments in the case of slowing down only.

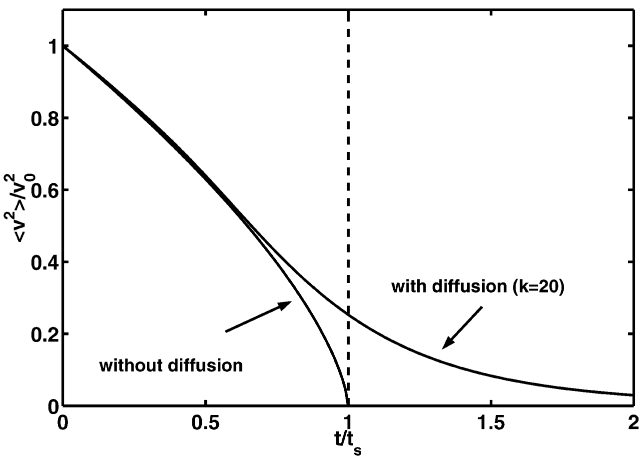

Fig. 2. Time variation of the second-order velocity moment for the cases without and with diffusion [as given by Eq. (10) for $k = \alpha/\beta = 20$].

to zero, the actual value of $\hat{t}_s$ depends crucially and artificially on the value of v_{th}. The reintroduction of the thermal velocity is physically neither consistent nor meaningful.

With the caveat that a single slowing-down time is too blunt a quantity to characterize such a complicated process as the evolution of the distribution function, in particular in the presence of diffusion, it seems that a natural and physically informative definition of the slowing down time t_s^* is to define it by considering the timescale for the initial time evolution. From the initial qualitative picture, we expect energy diffusion to affect the slowing-down time by the multiplicative factor $(1 - \beta/\alpha)^{-1}$, a prediction that will be confirmed by our subsequent analysis.

On the other hand, it can be argued that a physically more natural way of writing the Fokker-Planck equation is the form given by Eq. (2) because it more clearly separates the friction and diffusion operators. Consequently, by redefining $\alpha \to \alpha - \beta = \tilde{\alpha} = nM\theta(1 - M)$, the characteristic slowing-down time would be defined by $t_s \to \tilde{t}_s = v_0^3/\tilde{\alpha}$, which has the advantage of being (formally) independent of the diffusion parameter β. Since the choice of formulation of the Fokker-Planck equation is a matter of taste, we have chosen (for easy comparison with the results of Refs. 1 and 2) to use the form given by Eq. (1).

IV. MOMENT ANALYSIS

As emphasized in Sec. III, the goal of the present analysis is to obtain, from a knowledge of the distribution function, information about physically important quantities that are expressed as certain moments of the distribution function like, for example, the mean energy of the beam particles. For moments where the weighting function $F(v, \mu)$ only depends on v, i.e., $F(v, \mu) = F(v)$, the relevant information can be obtained directly from the pitch-angle integrated form of the Fokker-Planck equation, which reads

$$\frac{\partial \bar{f}}{\partial t} = \frac{\partial}{\partial v}\left[\frac{\alpha}{v^2}\,\bar{f} + \frac{\partial}{\partial v}\left(\frac{\beta}{v}\,\bar{f}\right)\right] + N\delta(v - v_0)\delta(t) \quad (5)$$

where $\bar{f}$ is defined as

$$\bar{f} \equiv \int_{-1}^{+1} f(v, \mu, t)\, d\mu \ .$$

Thus, by solving Eq. (5), which is much simpler than the original Eq. (1), the desired moments can then be obtained as

$$\langle G(v) \rangle = \frac{\int G(v)\,\bar{f}(v,t)\,dv}{\int \bar{f}(v,t)\,dv} \ .$$

However, this approach can be pursued further by deriving a recursive set of ordinary differential equations for the velocity moments $\langle v^n \rangle$. After multiplying Eq. (5) by v^n and integrating over v between zero and infinity, we obtain

$$\frac{d}{dt}\,\langle v^n \rangle \approx -n[\alpha - (n-1)\beta]\langle v^{n-3} \rangle + v_0^n \delta(t) \ . \quad (6)$$

The moments have here been normalized with respect to the total (i.e., fast plus thermalized) number of particles. This coupled system of moment equations is approximate in the sense that slowing down has been assumed to be rapid enough to make it possible to neglect particle diffusion to velocities higher than v_0. Furthermore, Eq. (6) is valid for all n except $n = 0$, where the partial integration used in deriving it cannot be performed due to the nonvanishing (and unknown) thermalization flux at $v = 0$. Since we consider $\bar{f}$ as representing the distribution of the fast (nonthermalized) particles, the zero-order moment $\langle v^0 \rangle$ represents the ratio of fast particles relative to the total number of released particles, a quantity that cannot be determined from the iterative system. This fact together with the well-known feature of the coupling to successively higher-order moments makes it difficult to solve the coupled system. However, the system can easily be integrated to yield the initial time evolution of the moments in the form ($n > 0$)

$$\langle v^n \rangle \approx v_0^n \left\{1 - \frac{n}{3t_s}\left[1 - (n-1)\,\frac{\beta}{\alpha}\right]t\right\} \ . \quad (7)$$

Comparing with the diffusion-free expression for the initial time evolution of the moments, we are enticed to introduce the following generalization of the energy slowing-down time, namely,

$$t_s^* \equiv \frac{1}{1 - \dfrac{\beta}{\alpha}} \, t_s \ ,$$

in complete agreement with the dimensional estimate. Finally, we note that the third-order moment again has an especially simple form, and as long as the thermalization flux can be neglected (i.e., $\langle v^0 \rangle \approx 1$), it can be written as

$$\langle v^3 \rangle = v_0^3 [1 - 3(\alpha - 2\beta)t] \ .$$

In the analysis of Ref. 2, the slowing-down time was obtained as the sum of two complicated functions, one containing explicitly the thermal energy of the background particles and one that shows the same qualitative dependence on β as that found in the present analysis.

V. SOLUTION OF THE REDUCED FOKKER-PLANCK EQUATION

An important simplification obtained by going to the reduced pitch-angle integrated Fokker-Planck Eq. (5) is the fact that this equation allows an explicit analytical solution to be found as follows: Introduce new independent and dependent variables according to $z = v^{3/2}$ and $g(z,t) = v^{-p} f(v,t)$, where $p = (3-k)/2$ and $k = \alpha/\beta$. Equation (5) can then be transformed into an equation for $g(z,t)$, namely,

$$\frac{\partial g}{\partial t} = D\left(\frac{\partial^2 g}{\partial z^2} + \frac{1}{z}\frac{\partial g}{\partial z} - \frac{\nu^2}{z^2} g\right) + S(z,t) \ , \qquad (8)$$

where $\nu = (k+1)/3$, $D = 9\beta/4$, and $S(z,t) = N\delta(z^{2/3} - z_0^{2/3})\delta(t)$ $(z_0 = v_0^{3/2})$. The Green function corresponding to Eq. (8) is given by

$$G(\xi, z, t) = \frac{1}{2Dt} \exp\left[-\frac{z^2 + \xi^2}{4Dt}\right] I_\nu\left[\frac{z\xi}{2Dt}\right] H(t) \ ,$$

where I_ν is the modified Bessel function of order ν and $H(t)$ is the Heaviside step function. The solution for the pitch-angle averaged distribution function is then obtained as

$$\bar{f}(v,t) = \frac{3Nv_0^2}{4Dt}\left(\frac{v}{v_0}\right)^{(3-k)/2} \times \exp\left[-\frac{v^3 + v_0^3}{4Dt}\right] I_\nu\left[\frac{(vv_0)^{3/2}}{2Dt}\right] \ . \qquad (9)$$

This solution is approximate in the same sense as the coupled system of moment equations, i.e., it neglects the condition of vanishing flow at v_0 toward higer velocities. Expanding the solution for small times t, we find

$$\bar{f}(v,t) \simeq \frac{3N}{4}\sqrt{\frac{v_0}{\pi Dt}}\left(\frac{v}{v_0}\right)^{(3-2k)/4} \times \exp\left[-\frac{(v^{3/2} - v_0^{3/2})^2}{4Dt}\right] \ .$$

As expected, for small diffusion coefficients $k \gg 1$, the corresponding distribution function is a slowly spreading peaked distribution that slows down toward thermalization. An example of the evolution is given in Fig. 3.

The solution given by Eq. (9) can also be used to obtain analytical solutions for the time variation of the different velocity moments, namely,

$$\frac{\langle v^n \rangle(t)}{\langle v^n \rangle(0)} = \frac{\Gamma[\frac{1}{2}(1+\eta+\nu)]}{\Gamma[1+\nu]} \exp\left[-\frac{1}{\tau}\right] \times \tau^{-\frac{1}{2}(1+\nu-\eta)} {}_1F_1\left[\frac{1+\eta+\nu}{2}, 1+\nu; \frac{1}{\tau}\right] \ , \qquad (10)$$

where ${}_1F_1(a,b;x)$ denotes the hypergeometric function $\eta = (2n - k + 2)/3$, $\tau = 4t/(9t_d)$, and we have introduced the characteristic timescale for diffusive broadening t_d as $t_d \equiv v_0^3/(4\beta)$ [see the estimate given by Eq. (3)]. Expanding the hypergeometric function for large arguments x (i.e., small τ), we regain the short time evolution of the moments given by Eq. (7) as we should. Figure 2 gives an example of the variation of the second-order velocity moment as predicted by Eq. (10).

To lowest order in the diffusion constant β, we can obtain a simple approximate solution for $\bar{f}(v,t)$ using the Goodman method (see Ref. 5). For this purpose, we make a physically reasonable ansatz for the evolution of

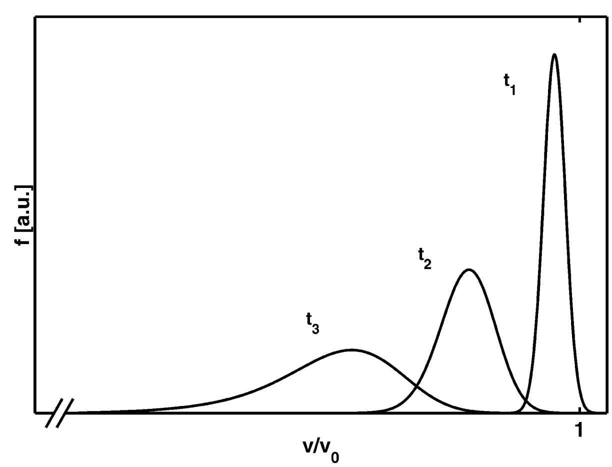

Fig. 3. An example of the evolution ($t_3 > t_2 > t_1 > 0$) of the pitch-angle integrated distribution function as given by the exact solution, Eq. (9). Note the qualitative agreement with the approximate Goodman solution, Eq. (11).

the distribution in the form of a Gaussian with varying amplitude, maximum velocity, and width, i.e.,

$$\bar{f}(v,t) = A(t)\exp\left[-\frac{(v-v_m(t))^2}{\Delta^2(t)}\right] \ , \qquad (11)$$

where the parameters $A(t)$, $v_m(t)$, and $\Delta(t)$ are functions to be determined from suitable moments of the original equation. As long as the number of thermalized particles remains small and the diffusive spreading is slow compared to the slowing down, we can approximate $A\Delta =$ constant and $v_m(t) \approx v_f(t)$. Using any of the higher-order moments, we obtain to lowest order in β that $\Delta(t) \approx v_0\sqrt{t/t_d}$, where the characteristic timescale for the diffusive broadening of the beam t_d appears naturally and again in good qualitative agreement with the dimensional estimate of Sec. II.

VI. PITCH-ANGLE SCATTERING DYNAMICS DURING SLOWING DOWN

Finally, we investigate the behavior during slowing down of quantities like the parallel and perpendicular beam temperatures, which depend on the pitch-angle scattering dynamics. Since we consider only situations where $\ln\Lambda \gg 1$, energy diffusion can be neglected in this analysis. The relevant Fokker-Planck equation is then

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left(\frac{\alpha}{v^2}f\right) + \frac{\gamma}{v^3}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu} + N\delta(v-v_0)\delta(\mu-1)\delta(t) \ . \qquad (12)$$

Eliminating time t in favor of the characteristic coordinate $\tau = t + v^3/(3\alpha)$ and introducing $F = f/v^2$ as a more convenient function, we obtain the simpler equation

$$\alpha\frac{\partial F}{\partial v} + \frac{\gamma}{v}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial F}{\partial\mu}\right] + N\delta(v-v_0)\delta(\mu-1)\delta(\tau - v^3/(3\alpha)) = 0 \ . \qquad (13)$$

Important and explicit information about the pitch-angle scattering dynamics during slowing down can be obtained analogously to the analysis in the previous sections by analyzing the evolution of certain physically important pitch-angle moments. To analyze the evolution of the parallel and perpendicular temperatures, it is convenient to study the moments

$$M_n(v,\tau) \equiv \frac{\alpha}{N}\int_{-1}^{1}\mu^n F\,d\mu \ .$$

After multiplying Eq. (13) with suitable weighting functions, the following ordinary equations are obtained:

$$\frac{dM_0}{dv} = -\delta(v-v_0)\delta(\tau - v_0^3/(3\alpha)) \ ,$$

$$\frac{dM_1}{dv} = \frac{2\gamma}{v}M_1 - \delta(v-v_0)\delta(\tau - v_0^3/(3\alpha)) \ ,$$

and

$$\frac{dM_2}{dv} = \frac{2\gamma}{v}(3M_2 - M_0) - \delta(v-v_0)\delta(\tau - v_0^3/(3\alpha)) \ .$$

These equations can readily be solved to give the time evolution of the moments during slowing down as follows:

$$M_0(t) = \text{constant} = 1 \ ,$$

$$M_1(t) = \left(\frac{v_f(t)}{v_0}\right)^{2\gamma/\alpha} \ ,$$

and

$$M_2(t) = \frac{1}{3}\left[1 + 2\left(\frac{v_f(t)}{v_0}\right)^{6\gamma/\alpha}\right] \ , \qquad (14)$$

where $v_f(t)$ is defined by Eq. (14). Clearly, the total energy $E = m\langle v^2\rangle/2$ of the particles still decays toward zero according to the result obtained in Sec. III, i.e., $E(t) = E(0)(1 - t/t_s)^{2/3}$. However, due to the pitch-angle scattering, the total energy is no longer confined to the parallel direction, and, for example, the mean parallel velocity $\langle v_\parallel\rangle$ decays according to

$$\langle v_\parallel\rangle = v_0\left(1 - \frac{t}{t_s}\right)^{1/3+2\gamma/3\alpha}$$

in complete agreement with the corresponding results obtained in Ref. 2 by integrating the expansion of the distribution function in terms of Legendre polynomials.

The results expressed by Eq. (14) also make it possible to describe the spreading of the beam in terms of its parallel and perpendicular temperatures and the concomitant anisotropic properties of the beam. For the parallel $T_\parallel \equiv m\langle(v_\parallel - \langle v_\parallel\rangle)^2\rangle/2$ and perpendicular $T_\perp \equiv m\langle(v_\perp - \langle v_\perp\rangle)^2\rangle$ temperatures, we obtain

$$\frac{T_\parallel}{E} = \frac{1}{3}\left[1 + 2\left(\frac{v_f}{v_0}\right)^{6\gamma/\alpha} - 3\left(\frac{v_f}{v_0}\right)^{4\gamma/\alpha}\right]$$

and

$$\frac{T_\perp}{E} = \frac{2}{3}\left[1 - \left(\frac{v_f}{v_0}\right)^{6\gamma/\alpha}\right] \ .$$

These formulas clearly show the evolution of the parallel and perpendicular temperatures as well as the evolution of the distribution function toward full isotropy as the particles slow down toward thermalization. At time $t = 0$, when $v = v_0$, both $T_\parallel$ and $T_\perp$ vanish, whereas as $t \to t_s$ and $v \to 0$, the ratios of the parallel and perpendicular temperatures to the full temperature approach the isotropic values, $\frac{1}{3}$ and $\frac{2}{3}$, respectively.

Although, as far as we know, no simple analytical solution of Eq. (12) describing the combined effects of slowing down and pitch-angle scattering is available, a simple approximate solution can again be found using the Goodman approach. For this purpose, we model the pitch-angle dependence with a function of the form

$$F(v,\mu) = A(v)\exp\left[\frac{\mu-1}{\Delta(v)}\right] \ .$$

Using the two moments $I_0(v)$ and $I_1(v)$ of $F(v,\mu)$, which are already known, we can obtain two relations, which determine the variation of $A(v)$ and $\Delta(v)$. For simplicity, we assume that $\Delta \ll 1$, i.e., we consider only the initial evolution before the beam has spread significantly in pitch angle. This assumption simplifies the evaluation of the moment integrals, and we obtain

$$A(v)\Delta(v) = \text{constant} \ ,$$

$$\Delta(v) \approx \left[1 - \left(1 - \frac{t}{t_s}\right)^{2\gamma/(3\alpha)}\right] \ .$$

To lowest order in t, we obtain for the initial spreading of the beam in pitch angle,

$$\Delta(t) \approx \frac{t}{t_p} \ ,$$

where the diffusion time t_p is defined by $t_p \equiv v_0^3/(2\gamma)$, again in good agreement with the qualitative estimate given in Sec. I.

VII. CONCLUDING REMARKS

This analysis has reinvestigated the problem of determining the dynamical properties of a beam of fast particles slowing down in a cold plasma under the combined influence of friction, energy diffusion, and pitch-angle scattering caused by small-angle collisions with background particles. This problem was previously analyzed in Refs. 1 and 2 using an exact, but cumbersome, solution of the concomitant Fokker-Planck equation. The present work emphasizes the physical understanding of the collisional processes and provides qualitative and approximate solutions for different aspects of the problem. In particular, the characteristic timescales for the different physical processes, which was one of the main points of the analysis in Ref. 2, have been clarified and expressed in simple and pregnant physical terms.

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