LOCALIZED FAST MAGNETOACOUSTIC EIGENMODES IN TOKAMAK PLASMAS

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ABSTRACT. An equation is derived for fast magnetoacoustic eigenmodes (FMEs) in a tokamak fusion plasma with an elongated cross-section. It is shown that edge localized fast magnetoacoustic waves (FMWs) with both positive and negative poloidal wavenumbers are possible, provided that the product $\hat{n}a^2$ exceeds a certain critical magnitude $N_{\rm cr}$, where $\hat{n}$ is the plasma density at a point near the plasma edge and a is the plasma radius. Fulfilment of this condition in various devices is analysed, which is important in explaining the fine structure of the spectrum of suprathermal ICE from tokamak plasmas. It is found that a small population of high energy ions with a non-equilibrium distribution function can significantly affect the FMEs.

1. INTRODUCTION

The toroidal drift of particles in a tokamak plasma significantly increases the growth rate of the cyclotron magnetoacoustic instability driven by high energy ions near the outer edge of a plasma [1–4]. As a result, the growth rate exceeds the bounce frequency of the fast ions even when the fast ion population n_α is very small compared with the background density n, i.e. $n_\alpha/n \leq 10^{-4}$ [1–3], which is the case in experiments. This provides a necessary condition for local destabilization of waves near the outer edge of a torus by particles with finite orbit width. The latter is important, in particular, for understanding the nature of the suprathermal ICE from tokamak plasmas, because it indicates that the instability referred to is the probable source of the ICE. According to Refs [1–4], another important effect of toroidal drift is that it is responsible for the fine structure (splitting into doublet) of the spectral lines of the ICE observed in JET [5]. The fine structure is a consequence of the fact that the instability growth rate is maximal for two values of the wave frequency, namely $\omega = l\omega_B \pm |\Omega_{\rm D}|$ (ω_B is the fast ion gyrofrequency, l is an integer, $\Omega_{\rm D} = mv_{\rm D}/r$, m is the poloidal wavenumber, r is the radial co-ordinate and $v_{\rm D}$ is the velocity of the toroidal drift of fast ions), provided that the modes with $m > 0$ and $m < 0$ are both excited.

The analysis of plasma stability in Refs [1–4] was carried out in a local approach. In fact, it relied on the results of studies in Ref. [6] where it was shown that, independently of the sign of the poloidal wavenumbers, fast magnetoacoustic waves (FMWs) can be localized near the plasma edge. However, a more exact equation for the fast magnetoacoustic eigenmodes (FMEs) derived in Ref. [7] contained a term which was dependent on the sign of m. Later, it was shown that this term can significantly affect the radial structure of the FMEs [8, 9]. Furthermore, it was found in Refs [8, 9] that in JET plasma the edge localized FMEs can have only one sign of m, whereas both $m > 0$ and $m < 0$ modes are possible in TFTR (where the ICE spectral lines are not split). The conclusions drawn were based on a condition which was obtained by making essential simplifying assumptions. In particular, the plasma cross-section was assumed to be circular and the plasma density on the magnetic axis, $n(0)$, was assumed to be related to the plasma density near the edge by means of a simple expression. Therefore, the analysis of Refs [8, 9] does not seem to be sufficient for definite conclusions. This stimulated the present article, where no assumption concerning $n(0)$ is made and a plasma with a non-circular cross-section is considered. In addition, the effects of fast ions on the FMEs are analysed.

The article is organized as follows. An equation for FMEs in a plasma with fast ions is derived in Section 2. At first, in Section 3, this equation is analysed neglecting fast ions. In particular, the conditions of existence of the edge localized FMEs are obtained and applied to JET and TFTR. The effects of fast ions are considered in Section 4. A summary of the results and a discussion are contained in Section 5. An equation for FMEs in a cold plasma with a non-circular cross-section is derived in Appendix A. In Appendix B an expression for a dielectric tensor component of alpha particles near the outer circumference of the torus is obtained.

2. EQUATION OF FAST MAGNETOACOUSTIC EIGENMODES IN THE PRESENCE OF FAST IONS

Let us begin with a derivation of an equation for the FMEs. We assume that the plasma contains a small population of fast ions and take into account the effects of the finite temperature of the bulk plasma, which can be important when the wave frequencies are close to the ion cyclotron harmonics. We consider fast magnetoacoustic waves with $k_\parallel \approx 0$ and $k_r \ll k$ ($k_\parallel$ and k_r are the longitudinal and radial wavenumbers, respectively), which implies that $k \approx |m|/r$. The modes with these wavenumbers can be localized near the plasma edge, at least in a cold plasma [6], and they can be destabilized with the largest growth rate [1–4]. Concerning the wave frequency, we assume that $\omega \ll \omega_B$. Such waves are of interest because a doublet structure of the ICE spectral lines has been observed for $l \geq 3$.

Let us present a perturbed quantity $\tilde{X}$ as

$$\tilde{X} = X(r)\exp(-\mathrm{i}\omega t + \mathrm{i}m\theta - \mathrm{i}n\varphi) \tag{1}$$

where θ and φ are the poloidal and toroidal angles, respectively, and n is the toroidal wavenumber. Then, using the orthogonal co-ordinate system with the basis vectors $\boldsymbol{e}_r = \boldsymbol{\nabla} r$, $\boldsymbol{e}_\parallel = \boldsymbol{B}_0/B_0$ and $\boldsymbol{e}_b = \boldsymbol{e}_\parallel \times \boldsymbol{e}_r$ (B_0 is the unperturbed magnetic field) and taking $E_\parallel = 0$, we can write the following set of equations for the perturbed electromagnetic field:

$$\frac{ck_b}{\omega}B_\parallel + \varepsilon_{11}E_r + \varepsilon_{12}E_b = 0 \tag{2a}$$

$$\frac{\mathrm{i}c}{\omega}\frac{\partial B_\parallel}{\partial r} + \varepsilon_{21}E_r + \varepsilon_{22}E_b = 0 \tag{2b}$$

$$B_r = B_b = 0 \tag{2c}$$

$$k_b E_r + \frac{\mathrm{i}}{r}\frac{\partial}{\partial r}rE_b + \frac{\omega}{c}B_\parallel = 0 \tag{2d}$$

where $k_b = m/r$, ε_{ij} is the dielectric tensor and E_i and B_i are components of the electric and magnetic fields, respectively. It has been assumed that $\boldsymbol{k}_\perp$ is aligned with the axis labelled '2' and that CGS Gaussian units are used. The plasma cross-section was taken to be circular (the effects of non-circularity will be included later, see Section 3 and Appendix A). The set of Eqs (2a–d) was obtained by using the Maxwell equations and the equation $j_i = \sigma_{ij}E_j$ where σ_{ij} is the plasma conductivity tensor associated with the dielectric tensor by means of the relationship $\varepsilon_{ij} \cong 4\pi i\sigma_{ij}/\omega$, $|4\pi\sigma_{ij}/\omega| \gg 1$.

Eliminating the components of the electric field from Eq. (2), we obtain the following equation for the FMEs:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{(\mathrm{i}\varepsilon_{21}k_bB_\parallel + \varepsilon_{11}B'_\parallel)}{\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{21}}\right) + k_b\,\frac{\mathrm{i}\varepsilon_{12}B'_\parallel - \varepsilon_{22}k_bB_\parallel}{\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{21}} + \frac{\omega^2}{c^2}B_\parallel = 0 \tag{3}$$

where $B' \equiv \partial B/\partial r$. Now we express ε_{ij} as $\varepsilon_{ij} = \varepsilon^{\mathrm{c}}_{ij} + \varepsilon^{\mathrm{T}}_{ij} + \varepsilon^{\alpha}_{ij}$, where $\varepsilon^{\mathrm{c}}_{ij}$ describes the bulk plasma neglecting thermal effects, $\varepsilon^{\mathrm{T}}_{ij}$ is the contribution to the dielectric tensor due to the finite Larmor radius of the bulk ions and $\varepsilon^{\alpha}_{ij}$ is relevant to the fast ions. Then, taking into account our assumptions we conclude that $|\varepsilon^{\mathrm{c}}_{12}| \gg |\varepsilon^{\mathrm{c}}_{11}| = |\varepsilon^{\mathrm{c}}_{22}|$, $\varepsilon_{ij} + \varepsilon_{ji} \cong 0$ for $i \neq j$, $|\varepsilon^{\alpha}_{ij}| \ll |\varepsilon^{\mathrm{c}}_{ij}|$ and $|\varepsilon^{\mathrm{T}}_{ij}| \ll |\varepsilon^{\mathrm{c}}_{ij}|$. This considerably simplifies the equation of the FMEs.

In particular, when the bulk ions consist of one species we find that

$$\frac{\partial^2 B_\parallel}{\partial r^2} - \left[k_b^2\left(1 + \frac{\varepsilon^{\mathrm{T}}_{22} + \varepsilon^{\alpha}_{22}}{\varepsilon^{\mathrm{c}}_{22}}\right) - \frac{\omega^2 n_{\mathrm{e}}^2}{v_{\mathrm{A}}^2 n_i^2} + k_b\,\frac{\omega}{\omega_B}\,\frac{n'_{\mathrm{e}}}{n_i}\right]B_\parallel = 0 \tag{4}$$

where

$$\varepsilon^{\mathrm{c}}_{22} = -\frac{\omega_{\mathrm{p}}^2}{\omega^2} \tag{5a}$$

$$\varepsilon^{\mathrm{T}}_{22} = -\frac{\omega_{\mathrm{p}}^2}{\omega}\sum_{|l|\neq 1}\frac{l^2 I_l(a_i)\exp(-a_i)}{a_i(\omega - l\omega_B)} \tag{5b}$$

$n' \equiv \partial n/\partial r$, ω_{p} is the ion plasma frequency, v_{A} is the Alfvén velocity, $a_i = k_\perp^2 v_i^2/(2\omega_B^2)$, I_l is a modified Bessel function of the first kind of order l and $v_i = \sqrt{2T_i/M_i}$ is the thermal velocity of the bulk ions.

Equation (4) can be also used for a description of the FMEs in a DT plasma, in which case one should carry out the summation over ion species in Eq. (5b) and take

$$v_{\rm A}^2 = B^2/(4\pi n_i M_{\rm D}) \tag{6a}$$

$$n_i = n_{\rm D}[1 + 2n_{\rm T}/(3n_{\rm D})] \tag{6b}$$

$$\omega_B = e_i B/(M_{\rm D} c). \tag{6c}$$

The effect of impurities on the dielectric tensor was neglected in Eq. (4). This is justified when $Z_i^2 n_{\rm I}/n_i \ll M_{\rm I}/M_i$ (the subscript 'I' refers to impurities and Z is the charge number).

An expression for $\varepsilon_{22}^{\alpha}$ for fast ions with a strongly non-equilibrium distribution function is derived in Appendix B.

Note that when fast ions are absent and ε_{ij}^{T} is negligible, Eq. (4) coincides with the corresponding equation of Ref. [7] taken at $k_\parallel = 0$. This equation is not symmetric with respect to the sign of m because of the presence of the last term, which is negligible in the plasma core but important near the plasma edge.

3. EDGE LOCALIZED FAST MAGNETOACOUSTIC EIGENMODES IN A COLD PLASMA

In this section, we will analyse Eq. (4) neglecting both the thermal corrections and the presence of fast ions. In addition, we assume that $n_{\rm e}(r)/n_i(r) =$ const. Then Eq. (4) can be written as follows:

$$\frac{\partial^2 B_\parallel}{\partial r^2} = H B_\parallel \tag{7}$$

where

$$H = k_b^2 - \frac{\tilde{\omega}^2}{v_{\rm A}^2} + k_b \, \frac{\tilde{\omega}}{\omega_B} \, \frac{n_i'}{n_i} \tag{8}$$

and $\tilde{\omega} = \omega n_{\rm e}/n_i$ (later the tilde will be omitted). In spite of the fact that Eq. (7) is derived for a plasma with a circular cross-section, it can be used also for a plasma with an elliptical cross-section after simple changes, namely, $m \to m\sqrt{(1+\kappa^{-2})/2}$, $e_i \to e_i\sqrt{(1+\kappa^2)/2}$ and its left hand side must be multiplied by $(1+\kappa^{-2})/2$ (κ is the cross-section ellipticity), Appendix A.

We are interested in the localized waves which have a characteristic width of localization, $L_{\rm w}$, small compared with the plasma radius but large compared with the wavelengths. Therefore, we can apply the method of perturbation theory to Eq. (7) by following the approach of Refs [6–10]. We write $\omega = \omega_0 + \delta\omega$ with $\delta\omega \ll \omega_0$ and express H as

$$H \approx H_0 + \left.\frac{\partial H}{\partial r}\right|_{r_*,\omega_0} (r - r_*) + \frac{1}{2} \left.\frac{\partial^2 H}{\partial r^2}\right|_{r_*,\omega_0} \times (r - r_*)^2 + \left.\frac{\partial H}{\partial \omega}\right|_{r_*,\omega_0} \delta\omega \tag{9}$$

where $H_0 \equiv H(r_*, \omega_0)$ and r_* is the radius for which the wave amplitude is maximum, with ω_0 and r_* being determined by the following zero order equations:

$$H(r_*, \omega_0) = 0, \qquad \left.\frac{\partial H}{\partial r}\right|_{r_*,\omega_0} = 0. \tag{10}$$

Using Eqs (9) and (10) one can find a solution which represents the localized FMEs.

A necessary condition for the existence of the localized modes is that the set of equations (10) should have a solution. This solution in the absence of the last term in Eq. (8) was found in Ref. [6]. However, the last term cannot be neglected near the plasma edge where $n(r)$ is small whereas n'/n is large. Comparing the last term with the first one in Eq. (8), we conclude that when $m > 0$ and $(a - r_*)/a \ll 1$, $\omega \sim k v_{\rm A}$, then the equation $H_0(\omega_0) = 0$ can be satisfied only if $a\omega_B/v_{\rm A} \propto a\sqrt{n}$ is sufficiently large, a being the plasma radius. On the other hand, when $m < 0$, there is no restriction on plasma parameters. Therefore, the edge localized modes with both $m > 0$ and $m < 0$ exist provided that

$$a^2 \hat{n} > N_{\rm cr} \tag{11}$$

where $\hat{n}$ is the plasma density in a point close to the plasma edge and $N_{\rm cr}$ is a quantity which will be determined below. In the case of a plasma with a non-circular cross-section, either a should be replaced by the effective plasma radius, $a_{\rm eff}$, or the factor describing the cross-section non-circularity should be included in $N_{\rm cr}$ and then a in Eq. (11) will denote the plasma radius in, for example, the equatorial plane of the torus. In particular, when a plasma has an elliptic cross-section then $a_{\rm eff} = a\sqrt{(1+\kappa^2)/2}$ (this will be clear when we obtain the expressions for $N_{\rm cr}$, see Eqs (20) and (21) below).

To find $N_{\rm cr}$, it is sufficient to specify the plasma profile shape in the periphery region. Assuming $n = n(\zeta)$ where $\zeta = \zeta(r)$ and keeping only the first term in the expansion of $n(\zeta)$ in the Taylor series near $\zeta_a = \zeta(a)$, we obtain (for $n(a) = 0$)

$$n(\zeta) = n_\zeta'(\zeta - 1) = \hat{n}\, \frac{1 - \zeta}{1 - \hat{\zeta}} \tag{12}$$

where $\hat{n} \equiv n(\hat{\zeta})$, $\hat{\zeta}$ is an arbitrary point near the plasma edge and $n'(\zeta) \equiv \partial n/\partial\zeta|_{\zeta_a}$. Equation (12) describes any profile near the plasma edge but there is an uncertainty in the dependence of ζ on r. We take $\zeta = (r/a)^\nu$ where ν is equal to 2 or 3. This choice can provide acceptable profile characteristics for $r/a \geq 0.7$ where presumably the source of the ICE is located. Other magnitudes of ν are not suitable. Indeed, $\nu = 1$ does not describe the change of the radial derivative of $n(r)$ in the region considered, therefore it does not correspond to realistic profile shapes. The case of $\nu > 3$ is not suitable because then $1 - \zeta$ is not small for $r/a = 0.7$–0.8 and thus expansion into the series is not justified.

Note that the chosen profile coincides with that given by

$$n(r) = n(0)(1 - r^2/a^2)^\mu \tag{13}$$

when $\mu = 1$ and $\nu = 2$. On the other hand, when $\mu < 1$, Eq. (13) differs significantly from Eq. (12) predicting, in particular, $n' \to \infty$ for $r \to a$, which is never the case in experiments. Nevertheless, the use of Eq. (13) with $\mu < 1$ is also justified, because in certain cases it gives better fits to experimental profiles in the intermediate region, for example, in $0.6 < r/a < 0.8$.

Equations (10) yield ω_0 and the equation for r_* as follows (cf. [8, 9]):

$$\omega_0 = k(\kappa) v_{A*} \left[\frac{\sigma_m v_A n'}{2\omega_B(\kappa) n} + \sqrt{1 + \left(\frac{v_A n'}{2\omega_B(\kappa) n} \right)^2} \right]_{r_*} \tag{14}$$

$$2 + r\frac{n'}{n} - \sigma_m \frac{v_A}{\omega_B(\kappa)} \left(\frac{rn'}{n} \right)' \left(1 - \frac{n'}{n} \frac{2 + rn'/n}{(rn'/n)'} \right)^{1/2} = 0 \tag{15}$$

where $k(\kappa) = |m|\sqrt{(1+\kappa^{-2})/2}/r$, $\omega_B(\kappa) = eB\sqrt{(1+\kappa^2)/2}/(Mc)$ and $\sigma_m = \operatorname{sgn} m$. It follows from Eq. (14) that, depending on the sign of m, ω_0 can be larger or smaller than kv_A. Note that the frequency ω_0 can be expressed as

$$\omega_0 = k_*(\kappa) v_{A*} \left[1 - \frac{n'}{n} \frac{2 + rn'/n}{(rn'/n)'} \right]_{r_*}^{-1/2}. \tag{16}$$

Using Eqs (12)–(15), we obtain an equation for r_* in the form

$$\frac{\hat{v}_A}{a\omega_B(\kappa)} = \sigma_m F(r_*) \tag{17}$$

where $F \equiv F_1$, for $n(r)$ given by Eq. (12), is

$$F_1 = -\frac{(1-\hat{\zeta})^{-1/2}}{\nu^{3/2}} \frac{[2 - (\nu+2)\zeta_*](1-\zeta_*)^{3/2}}{\zeta_*^{1-1/\nu}[\nu - 2 + (\nu+2)\zeta_*]^{1/2}} \tag{18}$$

$\zeta_* = (r_*/a)\nu$, and $F \equiv F_2$, for $n(r)$ given by Eq. (13), is

$$F_2 = -\frac{(1-\hat{x})^{-\mu/2}}{2\mu\sqrt{1+\mu}} (x_*^{-1} - 1 - \mu)(1 - x_*)^{1+\mu/2} \tag{19}$$

where $x_* = r_*^2/a^2$. The function $F(r_*)$ grows monotonically from $-\infty$ at $r_* = 0$ to $F_{\max} > 0$ at a certain point $r_m < a$. This means that Eq. (17) is always satisfied for $m < 0$ but in the case of $m > 0$ it can be satisfied only when $\hat{v}_A/[a\omega_B(\kappa)]$ does not exceed $F_{\max}$. Thus, we obtain Eq. (11) with N_{cr} determined as follows:

$$N_{\mathrm{cr},1} = 215(1-\hat{\zeta}) \frac{2}{1+\kappa^2} \frac{Mc^2}{\pi e^2} \tag{20}$$

$$N_{\mathrm{cr},2} = 4(1+\mu)\left(\frac{2+\mu}{\mu}\right)^{2+\mu} (1-\hat{x})^\mu \frac{2}{1+\kappa^2} \frac{Mc^2}{\pi e^2} \tag{21}$$

where $N_{\mathrm{cr},1}$ and $N_{\mathrm{cr},2}$ correspond to $F_1(r)$ and $F_2(r)$, respectively. Now we present the condition for the simultaneous existence of localized modes with $m > 0$ and $m < 0$ in a deuterium plasma in a form convenient for practical use,

$$\hat{n}_{19} a_m^2 > 8.96(1-\hat{\zeta}) \frac{2}{1+\kappa^2} \tag{22}$$

$$\hat{n}_{19} a_m^2 > 0.17\left(\frac{2+\mu}{\mu}\right)^{2+\mu} (1+\mu)(1-\hat{x})^\mu \frac{2}{1+\kappa^2} \tag{23}$$

where $n_{19} \equiv \hat{n}/10^{19}$, $\hat{n}$ is the plasma density in m^{-3} taken at an arbitrary point $\hat{r}$ near the plasma edge and a_m, the plasma radius in the equatorial plane of the torus, is in metres.

It is of interest to consider whether the conditions obtained are satisfied in experiments on tokamaks. For this purpose we use experimental data from JET [5] and TFTR [11]. We find that Eqs (12) and (13), which do not take into account the Shafranov shift of the magnetic flux surfaces, fit the experimental profiles sufficiently well only in certain parts of the plasma cross-section. Nevertheless, we can draw qualitative conclusions. We take $n = 10^{19}$ m^{-3} for $R = 4$ m (R is the distance from the major axis of the torus) in JET and $n = 2\times10^{19}$ m^{-3} for $R = 3.4$ m in TFTR (see Fig. 7 in Ref. [5] and Fig. 4 in Ref. [11]). For TFTR we find that neither Eq. (22) nor Eq. (23) is satisfied. On the other hand, Eq. (22) is well satisfied but Eq. (23) is not satisfied when $\mu = 0.5$ for JET. Thus, in order to draw a definite conclusion for JET, one must carry out an additional analysis. Not only better modelling of experimental profiles is required, but also taking into account additional physical factors. In the next section, we consider the effects of alpha particles.

4. EFFECTS OF ALPHA PARTICLES ON FAST MAGNETOACOUSTIC EIGENMODES

In order to study the effects of suprathermal alpha particles, we use Eq. (4), which we present in the form of Eq. (7) with H given by (we take $\kappa = 1$)

$$H = k_b^2(1+h) - \frac{\omega^2}{v_A^2} + k_b\,\frac{\omega}{\omega_B}\,\frac{n'}{n} \tag{24}$$

where $h = (\varepsilon_{22}^{\alpha} + \varepsilon_{22}^{T})/\varepsilon_{22}^{c}$, with $\varepsilon_{22}^{\alpha}$ depending on the fast ion distribution function and ε_{22}^{T} being given by Eq. (5b). Then, as in Section 3, we can use Eq. (10) for finding the eigenfrequency and the radius where a mode is localized. Assuming that $h \ll 1$, we write ω_0 as

$$\omega_0 = \omega_{00} + \delta\omega_0 \tag{25}$$

where ω_{00} is the eigenfrequency in a cold plasma without fast ions and $\delta\omega_0$ is a correction due to alpha particles and the finite Larmor radius of the bulk ions, $\delta\omega_0 = O(h)$. Using the equation (which follows from $H(r_*, \omega_0) = 0$),

$$\omega_0^2 = k_{b*}^2 v_A^2[1 + h_*(\omega_0)] + k_{b*}v_{A*}^2\,\frac{\omega_0}{\omega_B}\,\frac{n_*'}{n_*} \tag{26}$$

we express $H(r, \omega_0)$ as

$$H(r,\omega_0) = k_{b*}^2\left[\frac{r_*^2}{r^2}\,(1+h) - \frac{n}{n_*}\,(1+h_*) + \frac{\sigma_m\omega_{00}}{k\omega_B}\left(1+\frac{\delta\omega}{\omega_{00}}\right)y(r)\right] \tag{27}$$

where

$$y(r) = \frac{r_*}{r}\,\frac{n'}{n} - \frac{n(r)}{n_*}\,\frac{n_*'}{n_*}. \tag{28}$$

Calculation of $H'|_{r=r_*}$ yields

$$H'(r,\omega_0) = k_{b*}^2\left[z_*'(1+h_*) + h_*' + \frac{\sigma_m\omega_{00}}{k\omega_B}\left(1+\frac{\delta\omega}{\omega_{00}}\right)y_*'\right] \tag{29}$$

where $y_*' \equiv \partial y/\partial r|_{r_*}$, $z_*' \equiv \partial z/\partial r|_{r_*}$

$$z(r) = \frac{r_*^2}{r^2} - \frac{n}{n_*}. \tag{30}$$

Note that the equation $z'(r_*) = 0$ determines the point r_*, which does not depend on σ_m (this point of the mode location was found in Ref. [6]). The function $z(r)$ has a minimum at this point, $z(r)$ being limited for larger magnitudes of r ($z(a) \le r_*^2/a^2$). On the other hand, the last term in Eq. (27), proportional to $\sigma_m y(r)$, is negative for $\sigma_m > 0$ and dominates near the plasma edge ($y(r)$ is a monotonically decreasing function, $y(a) = -\infty$). Therefore, when $\sigma_m > 0$, this term shifts the mode localization radius towards the plasma edge and leads to the appearance of N_{cr}.

In order to investigate the effects of alpha particles and finite ε_{22}^{T} on N_{cr}, let us introduce a point r_{*0} determined by $H'|_{r=r*} = 0$ in the cold plasma, i.e. a point determined by the following equation (we used Eq. (30)):

$$z_*' + \frac{\omega_{00}}{k\omega_B}\,y_*' = 0. \tag{31}$$

When $N > N_{cr}^0$, where N_{cr}^0 is N_{cr} in the cold plasma, Eq. (31) cannot be satisfied for modes with $m > 0$, because the second term dominates. This term is negative for $m > 0$. Therefore, alpha particles would provide the increase of N_{cr} if their contribution to Eq. (29) at the radius $r = r_{*0}$ were positive. Taking Eqs (30) and (31) into account, this condition can be written in the following form:

$$H'(r,\omega_0)|_{r_{*0}} = k_{b*}^2\left[z'h\left(1 - \frac{\delta\omega}{\omega_{00}h}\right) + h'\right]_{r_{*0}} > 0. \tag{32}$$

Here ω_{00} is given by Eq. (14), and $\delta\omega_0$ can be found from the equation $H(r_*, \omega_0) = 0$ and Eq. (26) as

$$\delta\omega_0 = 0.5kv_Ah\Bigg/\sqrt{1+\left(\frac{n'}{n}\,\frac{v_A}{2\omega_B}\right)^2}\,\Bigg|_{r_{*0}}. \tag{33}$$

Further analysis requires that the distribution function of the fast ions F be specified. We consider equilibrium and strongly non-equilibrium distribution functions, which represent limiting cases. Namely, we consider:

(a) Maxwellian ions with a temperature greatly exceeding that of the bulk plasma,

(b) Ions with an anisotropic and non-monotonic velocity distribution described by

$$F(\boldsymbol{v}) = \frac{1}{2\pi^{3/2}v_0v_2}\exp\left(-\frac{(v_\parallel - v_1)^2}{v_0^2}\right)\delta(v_\perp - v_2). \tag{34}$$

Note that Eq. (34) was used for the interpretation of the ICE spectrum in experiments with DT plasmas in JET and TFTR [12, 13]. This function differs from the distribution function of alpha particles in JET, which was calculated in Ref. [14] and which can be approximated as $f = f_1(v)f_2(\chi)$, where $\chi = v_\parallel/v$. However, both the distribution function of Ref. [14] and that of Eq. (34) represent very anisotropic distributions. On the other hand, calculation of the Hermitian part of the alpha particle dielectric tensor with the realistic distribution function of Ref. [14] is rather difficult. These circumstances justify the use of $F(\boldsymbol{v})$ given by Eq. (34).

Let us compare the contribution to the permeability tensor due to the finite Larmor radius of the bulk ions and that due to the presence of a suprathermal ion population. If the alpha particles were Maxwellian and no toroidal effects were taken into account, then we could use Eqs (5a, b) not only for the bulk plasma but also for the alpha particles by changing a_i to a_α and ω_p to $\omega_{\mathrm{p}\alpha}$. In this case, assuming $\omega \approx kv_\mathrm{A} \approx l\omega_B$, we obtain $a_i \approx l^2\beta_i/2 \ll 1$ ($\beta_i = 8\pi n_i T_i/B^2$) for a near edge plasma with $\beta_i \approx 10^{-3}$, whereas $a_\alpha = l^2 v_\alpha^2/(2v_\mathrm{A}^2) \gg l$. This leads to very different magnitudes of $I_l(a_i)$ and $I_l(a_\alpha)$. As a consequence, $\varepsilon_{ij}^\alpha \gg \varepsilon_{ij}^\mathrm{T}$, even for a very small number of fast ions, the difference increasing with l. In particular, we find that, when $n_\alpha/n \sim 10^{-4}$, the alpha particle contribution dominates for $l \geq 3$. The contribution of fast ions to the permeability tensor becomes even larger when their distribution is non-equilibrium. Indeed, assuming $|\omega - l\omega_B| \sim |\Omega_\mathrm{D}|$ (a condition for the maximum growth rate of the instability [1–3]) and taking ε_{22}^α given by Eq. (53) in Appendix B, we obtain an additional large factor of the order of lv_α^2/v_0^2. Thus, the effects of the finite Larmor radius of the bulk ions on the structure of the FMEs can be neglected.

In contrast to this, the fast ions may significantly affect the FMEs. To demonstrate this, we give the following estimate for h obtained with the use of Eq. (53):

$$h \approx \sigma_m \frac{n_\alpha}{n} \frac{l^2 J_l^2(\xi_2)}{2\xi_2^2} \frac{v_2^4}{v_0^2 v_1^2} \frac{R}{\rho_2} \frac{\omega}{\omega_B} \tag{35}$$

where $\xi_2 = k_\perp v_2/\omega_B$ and $\rho_2 = v_2/\omega_B$. We observe that $h > 0$ and $\operatorname{sgn} h' = \operatorname{sgn}(n_\alpha r^3/n)'$ for $m > 0$, $\xi_2 \geq l$. When the profile shape of both the bulk ions and the alpha particles is described by Eq. (12) then $n_\alpha(r)/n(r) = n'_\alpha(a)/n'(a)$ and $h' > 0$. However, in general, the sign of h', and thus the sign of the right hand side of Eq. (32), can be either positive or negative. The magnitude of h is sensitive to the parameters in Eq. (34). For instance, taking $n_\alpha/n = 10^{-4}$, $R/\rho_2 = 30$ and $v_2/v_\mathrm{A} = 1.5$, we obtain $h \approx 0.1$ for $v_2/v_0 = 10$, $v_2/v_1 = 2$ (such a choice is in qualitative agreement with the distribution function calculated in Ref. [14]), and $h \approx 1.8$ for $v_2/v_0 = 20$, $v_2/v_1 = 4$ (these parameters were used in Ref. [12]). It is clear that in the latter case the effect of fast ions is so strong that it cannot be studied perturbatively.

5. SUMMARY AND CONCLUSIONS

Our analysis shows that edge localized FMEs with both positive and negative poloidal wavenumbers m exist, provided the effective number of plasma particles is sufficiently large. Namely, the following condition must be fulfilled:

$$\hat{n}a^2 > N_\mathrm{cr} \tag{36}$$

where a is the plasma radius in the equatorial plane of the torus and $\hat{n}$ is the plasma density at an arbitrary point $\hat{r}$ close to the region where the wave amplitude is expected to be maximum (e.g., $0.7 \leq r/a \leq 0.9$). N_cr depends on many plasma characteristics, in particular, it depends on the shape of the plasma cross-section, the plasma density profile shape at the periphery and the presence of fast ions. In addition, it depends on the chosen point $\hat{r}$. However, this dependence is relatively weak; therefore N_cr cannot vary by more than a factor of 2.

Equation (36) represents a condition for the splitting into a doublet of the spectral lines of the ICE by providing the possibility of a cyclotron magnetoacoustic instability with wave frequencies $\omega = l\omega_B \pm |\Omega_\mathrm{D}|$. Therefore, using Eq. (36) and on the basis of the theory given in Refs [1–4], we can predict that the spectral line of the ICE will be split into a doublet in future reactor tokamaks, for example, in ITER.

It seems that with the use of Eq. (36) one can also explain why the ICE spectral lines are split in JET but single in smaller tokamaks such as TFR and even in TFTR and JT-60U. However, we should note that the parameters of existing large tokamaks are such that Eq. (36) is either satisfied or not satisfied marginally. Because of this, in order to draw a definite conclusion concerning them, a special numerical simulation is required, which would include a description of the behaviour of fast ions near the edge. Furthermore, neither this article nor previous ones take into account the effects of toroidicity in the equation for FMEs with the $\operatorname{sgn} m$ dependent term, and thus development of the theory in this direction is required.

Finally, we hope that this work will stimulate new experiments on ICE, aimed at finding the sign of the poloidal wavenumbers of destabilized waves, which will verify the theoretical predictions and, thus, contribute to understanding the nature of ICE. According to Refs [1–4], the spectral lines of the ICE must be split into a doublet in those experiments where modes rotate in both the positive and the negative poloidal directions. On the other hand, by changing the profile characteristics near the plasma edge one can violate Eq. (36), which will lead to a significant change in the ICE spectra and, probably, to the merging of doublets into single lines.

Appendix A

EQUATION OF FAST MAGNETOACOUSTIC EIGENMODES IN A COLD PLASMA WITH NON-CIRCULAR CROSS-SECTION

Let us consider a cold plasma with an arbitrary cross-section that contains only one ion species. We introduce co-ordinates x^1, x^2, x^3, where x^1 is a radial co-ordinate such that $x^1 = \text{const}$ on a flux surface, $x^2 = \theta$ and $x^3 = \varphi$ are angular co-ordinates varying in the interval $(0, 2\pi)$ along the small and the large azimuth of the torus, respectively. Then, using the equations

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\,\boldsymbol{j}, \quad j_i = \sigma_{ij} E_j$$

and assuming that $X \propto \exp(-\mathrm{i}\omega t + \mathrm{i}m\theta - \mathrm{i}n\varphi)$, where X is a perturbed quantity, we can express E_1 and E_2 through B_3 as follows:

$$E_1 = \frac{c}{4\pi R^2 \sigma_{12}} \left(\mathrm{i}m\,\frac{\sigma_{22}}{\sigma_{12}}\,\sqrt{g}\,g^{22} B_3 + R\,\frac{\partial B_3}{\partial r} \right) \quad (37a)$$

$$E_2 = \frac{c}{4\pi R^2 \sigma_{12}} \left(\mathrm{i}m\,R B_3 - \frac{\sigma_{11}}{\sigma_{12}}\,\sqrt{g}\,g^{11}\,\frac{\partial B_3}{\partial r} \right). \quad (37b)$$

Here E_i and B_i $(i = 1, 2, 3)$ are the co-variant components of $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively, $g \equiv \det g_{ij}$ and g_{ij} is the metric tensor. An additional equation connecting E_1 and E_2 with B_3 results from

$$\boldsymbol{\nabla} \times \boldsymbol{E} = \mathrm{i}\,\frac{\omega}{c}\,\boldsymbol{B}$$

and can be written as

$$\frac{\partial E_2}{\partial r} - \mathrm{i}m E_1 = \mathrm{i}\,\frac{\omega}{c}\,\sqrt{g}\,\frac{1}{R^2}\,B_3. \quad (38)$$

Eliminating E_1 and E_2 using (37a, b) and (38), we obtain an equation for $B_3(r)$. We are interested in FMWs with large poloidal wavenumbers, $|m| \gg r/L_{\mathrm{W}}$, where L_{W} is the characteristic width of mode localization in the radial direction. In this case, the terms proportional to the first derivative of B_3 are negligible, and the derived equation can be presented in the following form:

$$g^{11}\,\frac{\partial^2 B_3}{\partial r^2} - \left(m^2 g^{22}\,\frac{\varepsilon_{22}}{\varepsilon_{11}} - \frac{\omega^2 \varepsilon_{12}^2}{c^2 \varepsilon_{11}} - \mathrm{i}m\,\frac{n'}{n}\,\frac{\varepsilon_{12}}{\varepsilon_{11}}\,\frac{R}{\sqrt{g}} \right) B_3 = 0. \quad (39)$$

Given that $\varepsilon_{11} = \varepsilon_{22}$ and $\mathrm{i}\varepsilon_{12}/\varepsilon_{11} = -\omega/\omega_B$, we obtain a flux surface averaged equation for $B_3(r)$ in the form

$$\overline{g^{11}}\,\frac{\partial^2 B_3}{\partial r^2} - \left(k_b^2 r^2 \overline{g^{22}} - \frac{\omega^2}{v_{\mathrm{A}}^2} + k_b\,\frac{n'}{n}\,\frac{\omega}{\omega_B}\,\frac{rR}{\sqrt{g}} \right) B_3 = 0 \quad (40)$$

where $\overline{g^{11}}$ and $\overline{g^{22}}$ are flux surface averaged quantities. In particular, in the case of a plasma with an elliptical cross-section, we can use the metric tensor given in, for example Ref. [15], which leads to

$$\overline{g^{11}} = r^2 \overline{g^{22}} = \frac{1 + \kappa^{-2}}{2}, \qquad \frac{\sqrt{g}}{rR} = \kappa \quad (41)$$

where κ is the elongation (ellipticity) of the flux surfaces defined as the ratio of major to minor radius of the ellipse, $\kappa(r)$ being taken constant. Thus, we arrive at the following equation:

$$\frac{1 + \kappa^{-2}}{2}\,\frac{\partial^2 B_3}{\partial r^2} = H B_3 \quad (42)$$

where

$$H = \tilde{k}_b^2 - \frac{\omega^2}{v_{\mathrm{A}}^2} + \tilde{k}_b\,\frac{n'}{n}\,\frac{\omega}{\tilde{\omega}_B} \quad (43a)$$

$$\tilde{k}_b^2 = \frac{1}{2}\,(1 + \kappa^{-2}) k_b^2 \quad (43b)$$

$$\tilde{\omega}_B = \tilde{e}B/(Mc) \quad (43c)$$

$$\tilde{e} = \sqrt{\frac{1 + \kappa^2}{2}}\,e. \quad (43d)$$

Note that in using the flux surface averaged equation for the FMEs we neglect the effects of the mode coupling associated with ellipticity. This is justified when the ellipticity is not too large. In order to obtain a corresponding condition, we take into account that

$$g^{11} = \overline{g^{11}} + \frac{1-\kappa^{-2}}{2}\cos 2\theta$$

$$g^{22} = \overline{g^{22}} - \frac{1-\kappa^{-2}}{2r^2}\cos 2\theta. \tag{44}$$

From Eqs (41) and (43a–d) it follows that ellipticity induced poloidal mode coupling can be neglected if

$$\frac{\kappa^2-1}{2(\kappa^2+1)} \ll 1. \tag{45}$$

Equation (45) is well satisfied in all cases of interest.

Appendix B

A DIELECTRIC TENSOR COMPONENT OF FAST IONS WITH NON-EQUILIBRIUM VELOCITY DISTRIBUTION IN THE PRESENCE OF THE TOROIDAL DRIFT

According to Eq. (4), the fast ions affect the radial structure of the FMWs mainly through $\varepsilon_{22}^{\alpha}$. Below we calculate this component of the dielectric tensor component assuming $k_r = k_\parallel = 0$ and taking the velocity distribution function in the form $F = f_1(v_\parallel)f_2(v_\perp)$, where

$$f_1 = \frac{1}{\sqrt{\pi}\,v_0}\exp\left(\frac{(v_\parallel - v_1)^2}{v_0^2}\right). \tag{46}$$

We proceed from the following expression (the superscript α is omitted):

$$\varepsilon_{22} = \sum_l 2\pi\frac{\omega_{\mathrm{p}\alpha}^2}{\omega}\frac{l^2\omega_B^2}{k_\perp^2}\int \mathrm{d}v_\perp\,\mathrm{d}v_\parallel\,\frac{J_l^2(\xi)}{\Omega}\frac{\partial F}{\partial v_\perp} \tag{47}$$

where

$$\Omega = \Delta\omega + \omega_{\mathrm{D}}, \quad \Delta\omega = \omega - l\omega_B$$

$$\omega_{\mathrm{D}} = \frac{k_b}{2\omega_B R}(v_\perp^2 + 2v_\parallel^2), \quad \xi = \frac{k_\perp v_\perp}{\omega_B}.$$

This equation takes into account the presence of toroidal drift and is relevant to the region where $\theta \approx 0$. It is obtained from the corresponding general expression given in Ref. [3].

Substituting $F = f_1(v_\parallel)f_2(v_\perp)$ into Eq. (47), we can express ε_{22} as

$$\varepsilon_{22} = -\sum_l 2\pi\frac{\omega_{\mathrm{p}\alpha}^2}{\omega}\frac{l^2\omega_B^2}{k_\perp^2}\int \mathrm{d}v_\perp f_2(v_\perp)\frac{\partial}{\partial v_\perp} \times\left(J_l^2\left\langle\frac{1}{\Omega}\right\rangle\right) \tag{48}$$

where $\langle 1/\Omega\rangle = \int_{-\infty}^{+\infty}\mathrm{d}v_\parallel\, f_1(v_\parallel)/\Omega$, and ω must be considered as a complex quantity with a small positive imaginary part when $1/\Omega$ has a singularity. Calculation of $\langle 1/\Omega\rangle$ with f_1 given by Eq. (46) yields

$$\left\langle\frac{1}{\Omega}\right\rangle = \frac{\omega_B R}{2|k_b u|v_0}\sum_\sigma\frac{\hat{Z}(z_\sigma)}{z_\sigma} \tag{49}$$

for $\sigma_{\Delta\omega}\sigma_m < 0$ and

$$\left\langle\frac{1}{\Omega}\right\rangle = \frac{1}{\Delta\omega + \omega_{\mathrm{D}}(v_\parallel = v_1)} \tag{50}$$

for $\sigma_{\Delta\omega}\sigma_m > 0$, where $\sigma = +1$ and -1, $\sigma_{\Delta\omega} = \operatorname{sgn}\Delta\omega$, $\hat{Z}(z_\sigma)$ is the plasma dispersion function,

$$z_\sigma = (u + \sigma v_1)/v_0$$

and

$$u^2 = -\frac{\omega_B R}{k_b}\Delta\omega - \frac{v_\perp^2}{2}.$$

In particular, when $f_2 = \delta(v_\perp - v_2)/(2\pi v_2)$, we obtain from (48)

$$\varepsilon_{22} = -\sum_l\frac{\omega_{\mathrm{p}\alpha}^2}{\omega}\frac{l^2\omega_B^2}{k_\perp^2}\frac{1}{v_2}\frac{\partial}{\partial v_2}\left(J_l^2\left\langle\frac{1}{\Omega}\right\rangle\right) \tag{51}$$

where $\langle 1/\Omega\rangle$ is given by Eqs (49) and (50) taken for $v_\perp = v_2$. In the case of $v_1 \gg v_0$, $k\rho \ll lR/\rho$ (ρ is the fast ion Larmor radius) and $|z_\sigma| \ll 1$, which is of interest for the resonant destabilization of FMWs by fast ions, we can write $\langle 1/\Omega\rangle$ as follows:

$$\left\langle\frac{1}{\Omega}\right\rangle = \frac{\omega_B R}{2|k_b|v_0}\left[\sigma_m\frac{2}{v_0}\left(\frac{v_1}{|u|} - 1\right) - \mathrm{i}\frac{\sqrt{\pi}}{|u|}\exp(-z_r^2)\right] \tag{52}$$

where $z_r^2 = (|u| - v_1)^2/v_0^2 < 1$. Note that the condition $z_r^2 < 1$ can be satisfied only when ω is close to one of the cyclotron harmonics. Retaining in Eq. (51) only the term where $\Delta\omega \approx l\omega_B$ and after substitution of Eq. (52) we obtain ε_{22} in the following form convenient for applications:

$$\varepsilon_{22} = -l^2\frac{\omega_{\mathrm{p}\alpha}^2}{\omega}\frac{v_2}{v_0}\frac{R}{\xi_2} \times\left\{\frac{\sigma_m}{v_0}\frac{\partial}{\partial\xi_2}\left[J_l^2\left(\frac{v_1}{|u_2|} - 1\right)\right] - \mathrm{i}\frac{\sqrt{\pi}}{2}\frac{\partial}{\partial\xi_2}\left[\frac{J_l^2}{|u_2|}\exp(-z_{r_2}^2)\right]\right\} \tag{53}$$

where $J_l = J_l(\xi_2)$, $\xi_2 = k_\perp v_2/\omega_B$, $u_2 = n(v_\perp = v_2)$ and $z_{r2} = z_r(v_\perp = v_2)$.

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