Modelling and measuring transport in fusion plasmas

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Göteborg, Sweden, 2009

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Abstract

In the present thesis we consider theoretical and experimental aspects of the turbulent transport which is a crucial issue in fusion plasma physics.

Experimental observations and gyrokinetic simulations show that collisions strongly influence the turbulent flux of particles. In this thesis we investigate the collisionality dependence of the quasilinear particle fluxes due to ion temperature gradient (ITG) and trapped electron (TE) modes. A semi-analytical, collisional model of electrostatic turbulence (COMET) has been developed, where collisions are modeled by the Lorentz operator. We point out that the form of the collision operator affects the collisionality scaling of particle flux. COMET has been benchmarked with the gyrokinetic code GYRO, and it is used to calculate quasilinear particle and energy fluxes and ITG mode stability thresholds. Closed analytical expressions are provided for the density and temperature responses without expansion in the smallness of magnetic drift frequency. We find that the temperature gradient threshold for stability is significantly affected by the electron-ion collisions for high enough logarithmic density gradients.

Alkali beam emission spectroscopy (BES) is widely used for the measurement of electron density and its fluctuations, contributing to the understanding of transport processes in the outer plasma regions. In the evaluation of density profile measurements the width of the diagnostic beam is often neglected, which might cause a non-negligible underestimation of the pedestal density. A de-convolution based correction algorithm has been introduced which estimates the emission density on the beam axis from a measured light profile allowing the use of the conventional one-dimensional density calculation methods.

Keywords: fusion plasmas, transport, microinstabilities, gyrokinetic equation, trapped electron response, ion temperature gradient mode, ITG stability, quasilinear flux, Lorentz operator, beam emission spectroscopy, de-convolution, electron density measurement
Publications


Other contributions
(not included in the thesis)


Contents

Abstract iii
Publications v
Acknowledgements ix

1 Introduction 1

2 Turbulent transport and microinstabilities 5
  2.1 Gyrokinetic description .......................... 6
  2.2 Ballooning formalism ............................. 9
  2.3 Quasilinear fluxes ............................... 11
  2.4 Microinstabilities ............................... 13
     2.4.1 Ion temperature gradient mode .............. 17
     2.4.2 Trapped electron mode ...................... 18
  2.5 Role of collisions in transport .................. 20

3 Beam emission spectroscopy 27
  3.1 Turbulence measurements ........................ 32
  3.2 Electron density measurements .................. 35

4 Summary 39

References 43

Included papers A-C 49
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Chapter 1

Introduction

The ever increasing energy demand of humanity, our finite non-renewable resources and the threatening extent of environmental pollution establish the need for a new, clean and large-scale energy source. One of the most promising candidates for this purpose is controlled thermonuclear fusion, which has been an intensely explored area for half a century.

Fusion utilizes the energy which is released as two light nuclei fuse together, and provides one of the main energy sources of the Universe. The reaction between the positively charged nuclei is obstructed by their Coulomb repulsion, so that bringing even the most feasible fusion process $[\text{D} + \text{T} \rightarrow ^4\text{He} (3.5 \text{ MeV}) + n (14 \text{ MeV})]$ to effect with reasonable efficiency (at achievable density) requires a temperature of $\sim 10^8 \, ^\circ\text{K}$, which implies that in laboratory conditions the fusion fuel has to be confined by some special means. At the same time, the extremely high temperatures giving rise to the difficulties of confinement also provide a possible solution, since the matter is then in almost fully ionized, plasma state which can be confined with a magnetic field.

As a consequence of the Poincaré–Hopf theorem [1], the only topology in three dimensions which has non-vanishing continuous tangent vector field is the torus. Therefore the most successful magnetic confinement fusion devices, the stellarator [2] and the tokamak [3] have toroidal magnetic geometry; their magnetic field lines trace out nested toroidal surfaces. The tokamak is axisymmetric and the twist of its magnetic field, which is necessary to avoid charge separation due to particle drifts, is maintained by a current driven inductively in the plasma. According to its relative simplicity, both its theory and technology has developed relatively rapidly, so that the fusion triple product $n_i T_i \tau_E$, which is the
main indicator of fusion performance, has doubled approximately every two years since the mid-1950’s ($n_i$ and $T_i$ are the ion density and temperature and $\tau_E$ is the characteristic time of the energy confinement). The stellarator has no net toroidal current, which has advantages – the possibility of continuous operation, no current driven instabilities and consequently no Greenwald density limit [4]. However, due to the lack of toroidal symmetry, its magnetic geometry has to be optimized to avoid drift losses from local magnetic wells. The helical structure of the magnetic field is produced by a complex magnetic coil system. Since the sufficient computational capacity for the optimized design of such complicated, inherently three-dimensional system has been achieved only in the recent years, the stellarator has lagged behind the tokamak concept, in terms of confinement.

On the road towards controlled thermonuclear fusion it was a notable event when, in 1997, the largest current fusion experiment, JET (Joint European Torus) produced 16 MW of fusion power – 65% of the input power [5]. The “next step” will be an even bigger tokamak experiment, ITER, which is under construction at the present time. Its goal is to demonstrate fusion energy production on a commercial scale by producing 500 MW fusion power from 50 MW input power [6]. Considerable knowledge about the behavior of fusion plasmas has been accumulated in recent decades that enabled the design of this experimental fusion reactor. However, the design is partly based on extrapolation using empirical scaling laws [7], since a comprehensive description of the transport processes determining the performance still does not exist due to the complexity of the problem.

For efficient fusion energy production the energy transport through the magnetic surfaces should be minimized, and at the same time, to maintain the ”burning” plasma, the particle transport has to be kept under control. In addition to the ubiquitous but tolerable level of diffusive collisional transport, the major part of the transport is due to convective fluxes associated with plasma turbulence. This turbulence is driven by various kinds of small-scale, low-frequency unstable modes, microinstabilities, due to density or temperature gradients [8]. Even hydrodynamic turbulence is a complex unresolved problem, and considering several fluid species coupled through electromagnetic, friction and energy exchange effects, it is not surprising that there is no general theory of plasma turbulence. Understanding the turbulent transport is one of the most important theoretical issues of magnetic confinement fusion.
The complexity of the problem — nonlinear coupling between the different modes, turbulent cascades through the different spatial scales, nonlinear self-regulation — almost makes analytical treatment impossible, although there are methods such as renormalization, quasilinear and mixing-length approaches that have been used — with limited success. It seems that one has to resort to non-linear fluid or kinetic simulations to obtain an overall picture of turbulent transport; accordingly, several fluid and kinetic simulation codes have been developed in the recent decades. However, it is possible, and rather important, to investigate the elementary properties and different parametric dependences of the microinstabilities.

The drive of turbulence in the innermost plasma region, the core, is typically dominated by ion temperature gradient (ITG) [9–11] and trapped electron (TE) [12–14] modes, but the electron temperature gradient (ETG) [15], micro-tearing modes, together with current diffusive ballooning and neoclassical tearing modes might also play some role [16–18]. Here, the level of the density and temperature fluctuations is only a few percent of the corresponding equilibrium quantities, while, in the edge and scrape-off layer (SOL) they can be comparable. In the latter, outer plasma regions mainly electrostatic fluid instabilities — driven by gradients in pressure, current or resistivity — dominate the turbulence.

To achieve reactor relevant experiments it is essential to understand and (to some extent) control the turbulent transport. It was discovered that applying sufficiently high auxiliary heating, a rapid spontaneous transition to an improved confinement mode — the so called high confinement- or H-mode — can be obtained, which involves a transport barrier formation at the plasma edge [19]. Due to the reduced transport at this edge transport barrier (ETB) the pressure gradient might exceed a certain magnetohydrodynamic stability limit, which leads to abrupt, quasi-periodic, burst-like ejection of a considerable part of the energy stored in the transport barrier region. These instabilities are called ELMs (edge localized modes) and can cause intolerable damage in the plasma limiting elements or at least contribute to their deterioration, but on the other hand they can be useful for helium ash removal [20].

For the deeper understanding of these complex phenomena determining the transport, strong interaction between theoretical and experimental work is needed. Due to the extremely high temperatures of fusion plasmas, the vast majority of diagnostic methods are not based on di-
rect physical contact between the measuring device and the plasma. A routinely used method of measuring electron density profiles and density fluctuation at the edge, SOL or outer core regions is alkali beam emission spectroscopy (BES) [21,22]. It is based on the observation of a high energy $10 - 100$ keV collimated neutral beam injected into the plasma. The photons – emitted in the spontaneous de-excitation of collisionally excited beam atoms – carry information on the distribution of plasma parameters determining the beam evolution.

The collisions also lead to ionization of the beam atoms, resulting in beam attenuation, which restricts the diagnostic to the outer plasma regions; accordingly it is well suited to turbulence measurements there as well as to the investigation of sheared flows in the edge (which play a role in the ETB formation), ELMs, geodesic acoustic modes and other phenomena affecting the transport. The spatial resolution of BES ($\sim 5$ mm at 40 keV beam energy [23]) is comparable with the distance covered by a beam atom during the characteristic time of its de-excitation. The time resolution of a fluctuation measurement ($\sim \mu$s) is mainly limited by the photon statistics, which is determined by the beam current (ion optics) and the efficiency of the observation system.

The remainder of the thesis is organized as follows. In the first part of Chapter 2, after a short introduction to the turbulence driven transport, the basics of the gyrokinetic description of magnetized plasmas are given, together with the ballooning formalism. Then the physical mechanism behind the quasilinear fluxes is presented. Microinstabilities are discussed with emphasis on the ion temperature gradient and the trapped electron modes. Finally, the role of collisions in transport, in particular in turbulent transport, is discussed. Chapter 3 concerns different possible applications of beam emission spectroscopy. After a general overview of the neutral beam diagnostics, short introduction to the theory of the BES turbulence and density profile measurements is given in Sections 3.1 and 3.2. Finally, in Chapter 4 the included papers are summarized.
Chapter 2

Turbulent transport and microinstabilities

It was recognized from early fusion experiments that the heat transport across the flux surfaces cannot be explained assuming only classical transport due to Coulomb collisions of plasma particles. The collisions give rise to diffusive transport characterized by the classical particle diffusion coefficient $D_c \sim \nu_{ei} \rho_{e}^2$, which is the same for both particle species in pure hydrogen plasma, and thermal diffusivities $\chi_\alpha \sim \nu_{\alpha\alpha} \rho_{\alpha}^2$, where $\nu_{\alpha_1\alpha_2}$ is the $\alpha_1$-$\alpha_2$ collision frequency, $\rho_\alpha$ is the gyro radius and the index $\alpha$ denotes the different particle species (conventionally $e$ – electron, $i$ – bulk ion). When the theory is extended to toroidal systems the step lengths are replaced by the width of the trapped particle orbits, and the effective collision frequency for particle de-trapping takes the role of the collision frequency, so that the resultant neoclassical transport levels exceed the classical ones by a factor of $q^2 \epsilon^{-3/2}$ [24], where $q = \langle \mathbf{B} \cdot \nabla \chi / \mathbf{B} \cdot \nabla \varphi \rangle_\Psi$ is the safety factor, $\epsilon$ is the ratio of the minor and major radii, $\mathbf{B}$ is the magnetic field, $\chi$ and $\varphi$ are the poloidal and toroidal coordinates, and we have denoted the flux-surface average by $\langle \cdot \rangle_\Psi$. Unfortunately, even this theory is not enough to reproduce the experimentally found rates and parametric dependences of transport.

As will become clear in the sequel, turbulent flows of the plasma referred traditionally as anomalous transport account for the major part of the particle and heat fluxes. The turbulence is driven by different drift-type microinstabilities that are destabilized by the inhomogeneities of plasma parameters.

A rough estimate of the turbulent transport can be obtained using
a simple random walk estimate for step length $L_C$ and step time $\tau_C$. If the turbulence is caused by drift waves, the time scale is comparable to the diamagnetic drift frequency $\omega_{*\alpha} = -k_\theta T_\alpha / e_\alpha B L_{n\alpha}$, where $k_\theta$ is the poloidal wave number $B$ is the magnetic field strength, $T_\alpha$, $e_\alpha$ and $L_{n\alpha}$ are the temperature, the charge, and equilibrium density scale length of species $\alpha$. The spatial scale is the gyro radius $L_C \sim \rho_\alpha$, thus the transport coefficient follows the gyro-reduced Bohm (or simply “gyro-Bohm” [25]) scaling

$$D_\perp \sim L_C^2 / \tau_C \sim (\rho_\alpha / L_T) T / e B.$$ \hspace{1cm} (2.1)

The level of transport is determined by the saturation amplitude of the perturbed quantities, and therefore the assumption of different saturation mechanisms leads to different diffusivities. One widely used approach is the mixing-length estimate [28], which balances the drift wave frequency $\sim \omega_{*\alpha}$ against the $E \times B$ nonlinearity leading to $D_\perp \sim \gamma / k_\perp^2$, where $\gamma$ is the linear growth rate of the most unstable mode. This is equivalent to a “wave breaking” picture, where the fluctuation amplitude saturates when the gradients of the perturbed quantities grow to the level of the equilibrium gradients

$$\nabla \hat{n} \sim \nabla n \Rightarrow k_\perp \hat{n} \sim n / L_n.$$ \hspace{1cm} (2.2)

Assuming a Boltzmann relation between the density and potential perturbations, ($\hat{n}$ and $\hat{\phi}$ respectively), this implies $e \hat{\phi} / T \sim (k_\perp L_n)^{-1}$. Balancing the growth rate against the $E \times B$ nonlinearity – the so-called weak turbulence model [29] – gives $D_\perp \sim \gamma^2 / k_\perp^2 \omega_{*}.$

### 2.1 Gyrokinetic description

The kinetic description of plasmas is a consistent approach to describe turbulent transport based on first principles. It provides the governing equations of the phase-space distribution function $f$ of any particle species from the principle of particle conservation and Liouville’s theorem, which imply that $f$ must be constant along particle trajectories (if there are no sources or sinks of particles present);

$$\partial_t f + (d_t z) \cdot \partial_z f = 0,$$ \hspace{1cm} (2.3)

where $z = \{z_i\}_{i=1}^6$ denotes arbitrary – not necessarily Cartesian – phase space coordinates. Hereafter we use the notation $d_t = \frac{d}{dt}$ and $\partial_\lambda = \frac{\partial}{\partial \lambda}$
for any scalar or vector quantity \( \lambda \) (in particular for spatial coordinates \( \partial_x \equiv \nabla \)). For the \( \{z_i\}_{i=1}^6 \rightarrow (x, v) \) case Eq. (2.3) reads as

\[
\partial_t f + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \partial_v f = 0,
\]

(2.4)

where the acceleration is given by the Lorentz force \( \mathbf{a} = \frac{e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \), and the fields contain both the macroscopic and microscopic fields. The microscopic fields can be interpreted as field fluctuations present on spatial scales smaller than the Debye length \( \lambda_D = \sqrt{\varepsilon_0 k_B T/n_e e^2} \) being responsible for the Coulomb collisions. It is useful to separate off these fields from the macroscopic ones, which yields the Fokker-Planck equation

\[
\partial_t f + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \partial_v f = C(f),
\]

(2.5)

where \( f \) and \( \mathbf{a} \) are now ensemble averaged quantities, and the collisions are described by the collision operator \( C(f) \), which is in general a differential operator depending on distributions of other species as well (see details in Sec. 2.5). The Fokker-Planck operator [30] provides the most accurate model of binary Coulomb collisions, but there exist several more approximate collision operators of different complexity and accuracy. The most important feature of the collision operator is positive entropy production, so that it drives the system towards local thermodynamic equilibrium.

In magnetic confinement fusion we consider systems that are magnetized in the sense that the gyro-radius of any species is much smaller than the size of the device \( L \) and the scale length of the equilibrium quantities accordingly \( \rho/L = \delta \ll 1 \). All quantities can be averaged over the fast Larmor gyration, reducing the problem to five dimensions by eliminating the gyro-phase dependence. The gyrokinetic description allows for sharp variations of the perturbed quantities; the perpendicular scale lengths can be comparable to the ion or electron Larmor radius \( k_{\perp} \rho \sim 1 \), albeit the perturbations are highly elongated along the magnetic field lines \( k_{\parallel} L \sim 1 \).

In the present section we follow the reasoning of Ref. [31], and neglect the collisions (we set \( C(f) = 0 \) in Eq. 2.5). A general nonlinear gyrokinetic equation including collisions is derived by a recursive formulation in Ref. [32], which is valid for general magnetic geometries with large flows on the order of the ion thermal speed.

The fields and distributions – represented by \( Y \) – are separated to equilibrium and perturbed parts \( Y = Y_0 + \hat{Y} \) so that \( \hat{Y}/Y_0 = \Delta \ll 1 \) and \( \partial_t Y_0 = 0 \). The gyrokinetic theory uses the following phase space
coordinates: the guiding center position \( \mathbf{X} \), the magnetic moment with regard to the unperturbed magnetic field \( \mu = \frac{m v_{\perp}^2}{2 B_0} \), the energy \( U = \frac{mv_{\perp}^2}{2} + e \phi \) involving the full electrostatic potential \( \phi = \phi_0 + \phi \) and the phase angle \( \gamma \) defined by \( \mathbf{v}_{\perp} = v_{\perp} (\mathbf{e}_2 \sin \gamma + \mathbf{e}_3 \cos \gamma) \), if the orthogonal right handed coordinate basis is \( \{ b, e_2, e_3 \} \) with \( b = B / B_0 \).

The kinetic equation, according to (2.3) is

\[
\partial_t f + d_t \mathbf{X} \cdot \partial \mathbf{X} f + d_t \mu \cdot \partial \mu f + d_t U \cdot \partial U f + d_t \gamma \cdot \partial \gamma f = 0. \tag{2.6}
\]

Denoting the guiding center velocity by \( \mathbf{V} \) and applying the orderings \( V_0 \sim v_t + \mathcal{O}(\delta v_t) \), \( \dot{V} \sim \Delta v_t \), \( \dot{\mu}_0 \sim \omega_t \), \( \dot{\mu} \sim \Delta \Omega \mu \) and \( \dot{d}_t U \sim \Delta \omega_t U \), where \( \Omega \) is the cyclotron frequency, \( \omega_t = v_t / L \) is the transit frequency and \( v_t \) is the thermal velocity, (2.6) can be gyro-averaged

\[
\partial_t \bar{f} + \langle \mathbf{V}_0 + \dot{\mathbf{V}} \rangle \cdot \partial \mathbf{X} \bar{f} + \langle d_t \mu \rangle \cdot \partial \mu \bar{f} + \langle d_t U \rangle \cdot \partial U \bar{f} = 0, \tag{2.7}
\]

where \( \langle Y \rangle \langle \mathbf{X} \rangle \equiv \frac{1}{2 \pi} \oint d\gamma Y(\mathbf{X} + \rho) \) denotes the gyro-average, and in lowest order we neglected the gyro-phase dependent part of the distribution function \( \bar{f} = f - \bar{f} \sim \mathcal{O}(\delta) + \mathcal{O}(\Delta) \). The calculation of the gyro-averaged quantities is elaborate but straightforward; omitting the details we find that

\[
\dot{\mu}_0 = -\frac{\mu}{B_0} \mathbf{v} \cdot \nabla B_0 - \frac{mv_{\perp}}{B_0} (\mathbf{v} \cdot \nabla) b + \frac{e}{B_0} \mathbf{v}_{\perp} \cdot \mathbf{E} \tag{2.8}
\]

vanishes on a gyro-average, so that \( \langle d_t \mu \rangle \equiv \langle \dot{\mu}_0 + \dot{\mu} \rangle = \langle \dot{\mu} \rangle \) which is just

\[
\langle \dot{\mu} \rangle = \frac{e}{m B_0} v_{\perp} \left( (\hat{\mathbf{s}} \cdot \dot{\mathbf{E}}) - v_{\parallel} \langle \hat{\rho} \cdot \dot{\mathbf{B}} \rangle \right), \tag{2.9}
\]

where we have introduced the unit vectors \( \hat{\mathbf{s}} = \mathbf{v}_{\perp} / v_{\perp} \) and \( \hat{\rho} = \rho / \rho \) with \( \rho = (\mathbf{x} - \mathbf{X}) \). The gyro-averaged lowest order guiding center velocity is

\[
\langle \mathbf{V}_0 \rangle = \mathbf{v}_{\parallel} + \mathbf{v}_{D0}, \tag{2.10}
\]

with

\[
\mathbf{v}_{D0} = \mathbf{v}_{E0} + b \times (\mu \nabla B_0 + v_{\parallel}^2 \kappa), \tag{2.11}
\]

where \( \mathbf{v}_{E} = (\mathbf{E}_0 + \dot{\mathbf{E}}) \times b / B_0 = \mathbf{v}_{E0} + \mathbf{v}_{E} \) is the \( \mathbf{E} \times \mathbf{B} \) drift velocity and \( \kappa = b \cdot \nabla b \) is the field curvature. We refer to the second term in (2.11) as the magnetic drift velocity. The gyro-averaged perturbed velocity is

\[
\langle \dot{\mathbf{V}} \rangle = \frac{1}{B_0} \left( (\dot{\mathbf{E}} + \mathbf{v}_{\parallel} \times \dot{\mathbf{B}}) \times b - (\hat{\mathbf{s}} \hat{B}_{\|}) \right). \tag{2.12}
\]
Finally
\[ \langle d_t U \rangle = e \left( \langle \partial_t \hat{\phi} \rangle - \langle \mathbf{v} \cdot \partial_t \hat{A} \rangle \right). \] (2.13)
Suppressing the over-bar the lowest order gyrokinetic equation becomes
\[ \partial_t \hat{f} + \langle \mathbf{V}_0 \rangle \cdot \partial_X \hat{f} = -\langle \hat{\mathbf{V}} \rangle \cdot \partial_X f_0 - \langle d_t U \rangle \partial_U f_0. \] (2.14)
Note, that since \( U \) contains both \( \phi_0 \) and \( \hat{\phi} \), the full first order solution is \( \hat{f}(U_0) + e \hat{\phi} \partial_U f_0 \) where we refer to the first term as non-adiabatic and the second term as adiabatic (or Boltzmann) response.

We express the perturbed fields through the eikonal approximation
\[ \hat{Y}(\mathbf{x}) = Y_*(\mathbf{X}) e^{ik_{\perp} \cdot \mathbf{x}}, \] (2.15)
where \( Y_* \) and \( k_{\perp} \) are spatially slowly varying functions. The gyro-average is
\[ \langle \hat{Y} \rangle(\mathbf{X}) = e^{ik_{\perp} \cdot \mathbf{X}} Y_* \langle e^{ik_{\perp} \cdot \mathbf{r}} \rangle. \] (2.16)
The average can easily be evaluated in terms of the Bessel function of the first kind \( J_n \), using that
\[ J_n(z) = \frac{1}{2\pi} \int d\gamma e^{-in\gamma + iz \sin \gamma}. \] (2.17)
Thus we have
\[ \langle e^{ik_{\perp} \cdot \mathbf{r}} \rangle = J_0(k_{\perp} \rho), \]
\[ \langle \mathbf{s} e^{ik_{\perp} \cdot \mathbf{r}} \rangle = iJ_1(k_{\perp} \rho)(k_{\perp} \times \mathbf{b})/k_{\perp}, \]
\[ \langle \mathbf{r} e^{ik_{\perp} \cdot \mathbf{r}} \rangle = iJ_1(k_{\perp} \rho)k_{\perp}/k_{\perp}. \] (2.18)
Substituting back the explicit form of the gyro-averages into (2.14) and introducing \( Y_A = Y_* e^{ik_{\perp} \cdot \mathbf{X}} \) one finds that
\[ \partial_t \hat{f} + (\mathbf{v}_\| + \mathbf{v}_D) \cdot \partial_X \hat{f} = -[J_0(k_{\perp} \rho) \left( \partial_t \hat{\phi}_A - v_\| \partial_t A_{A\|} \right)] e^{i\phi_0} \]
\[ + (v_\perp/k_{\perp})J_1(k_{\perp} \rho) \partial_t B_{A\|} \left( e^{i\phi_0} + B_0^{-1}k \times b \cdot \nabla f_0 \right). \] (2.19)

### 2.2 Ballooning formalism

Any general plasma perturbation in a torus has to be periodic in both toroidal and poloidal coordinates, thus an elementary perturbation has the form \( \hat{Y}(r, \chi, \varphi) = \hat{y}_n(r, \chi) e^{i(m\chi - n\varphi)}, \) where the perturbed quantity depends on radius \( r \), the poloidal angle \( \chi \) and the toroidal angle \( \varphi \).
Chapter 2. Turbulent transport and microinstabilities

For small-scale perturbations \( 1 \ll m, n \in \mathbb{Z} \), and \( \hat{y}_n(r, \chi) \) is a slowly varying, \( \chi \)-periodic envelope. However, the radial dependence of the safety factor \( q \) is incompatible with the \( \chi \) periodicity, since a constraint that \( \hat{y} \) vanishes at the end of the basic \( \chi \) interval would be artificial.

It is convenient to consider the problem in the ballooning representation [33] which is appropriate for the description of mode structures characterized by short perpendicular and long parallel wavelengths when the magnetic shear \( s = (r/q)(dq/dr) \) is finite. Using an eikonal representation, the \( n \)th toroidal harmonic can be expressed as \( \hat{Y}_n(r, \chi, \varphi) = \hat{y}_n(r, \chi) e^{-in[\varphi-q(r)\chi]} \), which can further be written as

\[
\hat{Y}_n(r, \chi, \varphi) = \sum_{\theta_0} \sum_{j=-\infty}^{\infty} \hat{Y}_{B,n}(\chi + 2\pi j, \theta_0)e^{-in[\varphi-q(r)(\chi+2\pi j+\theta_0)]},
\]

(2.20)

where the ballooning function \( \hat{Y}_{B,n} \) depending on the extended poloidal angle \( \theta = \chi + 2\pi j \in ]-\infty, \infty[ \) has been introduced together with the ballooning angle \( \theta_0 \), which acts as linear eigenmode label [34]. The originally two-dimensional problem for \( \hat{y}_n(r, \chi) \) – with a periodicity condition in \( \chi \) – now reduces to a series of one dimensional calculations for \( \hat{Y}_{B,n}(\theta, \theta_0) \) with the much simpler condition \( \hat{Y}_{B,n}(|\theta| \to \infty, \theta_0) \to 0 \). In the limiting case of \( \rho_* \to 0 \), keeping \( N \) terms of the \( \theta_0 \) series, so that \( \theta_0 = \{2\pi l/N \mod 2\pi\}_{l=1}^{N} \), gives radially \( N\Delta \) periodic eigenmode solutions, where \( \Delta = (nq')^{-1} \) is the distance between the adjacent rational surfaces. The use of this expansion becomes apparent if we note that the most unstable mode can usually be calculated by considering only the \( \theta_0 = 0 \) term.

For a circular flux surface, axisymmetric, large aspect ratio \( (1 \ll R/a) \) equilibrium the linearized gyrokinetic equation for species \( \alpha \) reads in the ballooning representation as [35]

\[
\frac{v_{||}}{qR} \partial_\theta g_\alpha - i(\omega - \omega_{D\alpha}) g_\alpha - C(g_\alpha) = -i e_\alpha \int_0 T_\alpha (\omega - \omega^{*\alpha}_{\alpha}) J_0(z_\alpha),
\]

(2.21)

where we proceeded from Eq. (2.19) considering purely electrostatic perturbations \( \{A_{A||}, B_{A||}\} = 0 \) and finite collisionality, we set \( E_0 = 0 \), and used the notation \( g = \hat{f}_{B,n} \) and \( \phi = \hat{\phi}_{B,n} \). The time derivative is expressed in terms of wave frequency \( \partial_t \hat{Y} \to -i\omega \hat{Y} \). Conventionally \( \omega_{\alpha}^{*\alpha} = \omega_{\alpha} \left[ 1 + \left( x_\alpha^2 - \frac{3}{2} \right) \frac{L_{n\alpha}}{L_{T\alpha}} \right] \), with \( \omega_{\alpha} \) the diamagnetic frequency, \( x_\alpha \) is the velocity normalized to the thermal speed, \( L_{n\alpha} = -[\partial_r(\ln n_\alpha)]^{-1} \) and \( L_{T\alpha} = -[\partial_r(\ln T_\alpha)]^{-1} \) are the density and temperature scale lengths,
2.3. Quasilinear fluxes

\[ \omega_{D\alpha} = -k_\theta (v^2_\perp / 2 + v^2_\parallel) \left( \cos \theta + s \theta \sin \theta \right) / (\Omega_\alpha R) \]

is the magnetic drift frequency. The argument of the Bessel function being responsible for the finite Larmor radius effects is \[ z_\alpha = k_\theta v_\perp \alpha \sqrt{1 + s^2 \theta^2 / \Omega_\alpha}. \]

The equilibrium distribution \( f_{\alpha 0} \) is taken to be Maxwellian.

2.3 Quasilinear fluxes

The right-hand side of the gyrokinetic equation (2.21), which is proportional to the potential perturbation, acts as a source, so that the full solution of the gyrokinetic problem \( \hat{F}_\alpha \sim g_\alpha - e_\alpha \phi f_{\alpha 0} / T_\alpha \) is in general of the form \( n_e (x + iy) e^{\phi / T} \), where the functions \( x \) and \( y \) have a dependence on the velocity space variables and depend parametrically on the plasma parameter profiles. The density and temperature perturbations are calculated as \( \hat{n}_\alpha = \int d^3J_0(z_\alpha) \hat{F}_\alpha \) and \( \hat{T}_\alpha = \int d^3J_0(z_\alpha) v^2 \hat{F}_\alpha \), where the Bessel function appears in the back-transformation from guiding center to particle position coordinates. Thus the density and temperature responses \( \hat{n}_\alpha / n_\alpha \) and \( \hat{T}_\alpha / T_\alpha \) also have an imaginary part describing a phase difference between the potential perturbation and the perturbed quantity, giving rise to the quasilinear particle and energy fluxes across the magnetic surfaces. The potential perturbation \( \hat{\phi} \) corresponds to a perturbed drift velocity \( \hat{v}_E = b \times \nabla \hat{\phi} / B \) producing an ambiopolar particle flow. The flux surface average of this flow is the quasilinear particle flux \( \Gamma \) \[ \Gamma_\alpha = \Re \langle \hat{n}_\alpha \hat{v}_E^* \cdot \hat{r} \rangle \psi \] \[ (2.22) \]

where \( \hat{r} \) is the radial unit vector. Using that \( (b \times \nabla \hat{\phi}^*) \cdot \hat{r} = ik_\theta \hat{\phi}^* \), the particle flux can be written as

\[ \Gamma_\alpha = -\frac{k_\theta T_\alpha n_\alpha}{eB} \left| e\hat{\phi} / T_\alpha \right|^2 \Im \left\{ \frac{\hat{n}_\alpha / n_\alpha}{e\hat{\phi} / T_\alpha} \right\}, \] \[ (2.23) \]

where the over-bar denotes the flux surface average of the perturbed quantities. Similarly one finds that the energy flux is

\[ Q_\alpha = -\frac{k_\theta T^2_\alpha n_\alpha}{eB} \left| e\hat{\phi} / T_\alpha \right|^2 \Im \left\{ \frac{\hat{T}_\alpha / T_\alpha}{e\hat{\phi} / T_\alpha} \right\}. \] \[ (2.24) \]

It is important to note that if we assume only one singly charged ion species, and neglect the non-adiabatic electron response \( y \to 0 \), then
since $\Gamma \sim y$; the Boltzmann response has no imaginary part, and the ambipolarity holds ($\sum_{\alpha} e_{\alpha} \Gamma_{\alpha} = 0$), there will be no particle fluxes.

The mechanism of the quasilinear fluxes is illustrated in Fig. 2.1. The $E \times B$ flows indicated by blue arrows drift along the equipotential surfaces. Since the whole structure moves poloidally (gray arrow) with the diamagnetic velocity, and the maximum of the perturbed density lags behind the potential maximum – due to the phase shift between $\hat{n}$ and $\hat{\phi}$, – the density is apparently higher in the upward than the downward flow region; therefore a net particle flux appears across the flux surfaces due to the imbalance in the particle flows. The same picture holds for temperature perturbations and energy fluxes.

![Figure 2.1: Schematic picture on the mechanism behind the quasilinear fluxes of electrostatic microinstabilities. The perturbed potential – giving rise to $E \times B$ flows – is color density plotted; the perturbed density is contour plotted with dotted lines.](image)

In reality one should consider a whole spectrum of nonlinearly interacting modes, which means that the fluxes given in Eq. (2.23) and (2.24) should be summed up over the different modes characterized by $k_\theta$, however Eq. (2.23) is still valid. In most cases by taking only the most unstable mode, at least qualitatively accurate conclusions can be drawn.
2.4 Microinstabilities

The generation of fine-scale turbulence in plasmas is believed to be produced by microinstabilities [36, 37], i.e. instabilities which have wavelengths that are comparable to the ion or electron Larmor radii. To obtain an overall picture of turbulent transport, and for the calculation of turbulence saturation levels, nonlinear processes have to be taken into account. However it is useful to identify the possible drives and conditions of turbulent processes. Investigation of linear mode characteristics can provide estimates of stability thresholds and parametric dependences of turbulent fluxes.

Since the goal of fusion experiments is to sustain enormous temperature gradients (the \( \sim 4^\circ K \approx 4 \cdot 10^{-4} \) eV cryostat of a superconducting device is separated from the \( \sim 10 \) keV plasma core only by a few meters or less), the plasma is always far from thermodynamic equilibrium. In this non-equilibrium state the available free energy might be transferred to turbulent flows via instabilities. Drift waves are particularly important class of microinstabilities which have often been invoked as the main source of plasma turbulence. Dissipation through, e.g., collisions or kinetic resonances often plays an important role in the destabilization of the drift waves. We mainly focus on these type of instabilities classified as dissipative modes, while some others, the reactive ones, do not require dissipation – similarly to a wide range of magnetohydrodynamic (MHD) instabilities. A quite important class of instabilities – which is responsible for a major part of the transport – is predominantly electrostatic, although in the plasma core, where the kinetic pressure normalized to the magnetic pressure \( \beta = p/(B^2/2\mu_0) \) is not negligibly small, the electromagnetic modes might also play role [37]. In the present thesis we restrict our studies to electrostatic microinstabilities.

The microinstabilities have spatial scales that are typically much longer than the Debye length, in addition, they are slow instabilities compared to the plasma waves, so that the quasineutrality \( \sum_\alpha e_\alpha \hat{n}_\alpha = 0 \) can be shown to be a very good approximation. This does not mean that there are no electric fields involved, on the contrary, the most dangerous instabilities are electrostatic; but the amplitude of the electrostatic fluctuations requires extremely low charge separation. For this reason one must in general calculate the potential self-consistently from Poisson’s equation [38]. The quasineutrality condition, in turn, can be used to obtain a dispersion relation from the density responses of the different species.
The spatial and frequency scales of the microinstabilities that are thought to be accountable for the turbulence are indicated in Fig. 2.2 together with MHD and cyclotron waves. Clearly, the microinstabilities have much lower frequencies than the cyclotron frequencies ($\omega_{cc}, \omega_{ci}$), which allows for the use of gyro-averaged equations. The wavelengths are $10^{-10^4}$ times smaller than the size of the system, ranging from $\sim 10$ times the ion Larmor radius to the electron Larmor radius. In contrast to the MHD waves, this feature makes them somewhat less sensitive to the shaping effects of the geometry, thus analytical calculations often rely on the framework of a simple, circular cross section, large aspect ratio model.

The ion temperature gradient (ITG) mode driven by ion magnetic drift and transit resonances, which is recognized as the most important drive of turbulence (to be discussed in detail in Sec. 2.4.1 [9–11]), is characterized by $k_\theta \rho_i \lesssim 1$, but since the maximum growth rate in the turbulent spectrum usually corresponds to $k_\theta \rho_i \sim 0.2$, an expansion in the FLR parameter might be appropriate. The trapped electron mode (TE or TEM, see Sec. 2.4.2 [12–14]), appearing in the same wave number range $k_\theta \rho_i \lesssim 1$, is destabilized by electron magnetic drift resonances or collisional dissipation. The TE and ITG modes have a frequency range between the ion and electron bounce frequencies ($\omega_{bi}, \omega_{be}$) which are comparable with the corresponding transit frequencies ($\omega_{ti}, \omega_{te}$). This means that the trapped electrons bounce several times during a wave period; thus usually a bounce averaged electron gyrokinetic equation is used, in which the $v_\parallel/(qR) \partial_\theta g_e$ term is annihilated by the averaging operation. Trapping effects become important when the mode frequency is comparable to or lower than the bounce frequency, since then the particles are moving quickly enough along the field lines to sample the whole toroidal geometry during a mode period. Whereas for electron temperature gradient (ETG) and ITG modes the frequencies are higher than the electron/ion bounce frequencies, the corresponding trapped particle modes, namely, the trapped electron mode (TEM) and the trapped ion mode (TIM) [39], have frequencies that are lower than or comparable to the bounce frequencies, as it can be seen in Fig. 2.2. Since $\omega_{bi} \ll \omega_{ITG}$, the ion trapping is often neglected in ITG studies [40].

As an illustration of the drift wave phenomenon, from the quasineutrality condition one can derive the simplest possible electrostatic drift wave by assuming adiabatic electron response, and deriving the ion response from Eq. (2.21), neglecting the FLR effects ($J_0(z_i \to 0) \to 1$) and
Figure 2.2: Typical spatio-temporal scales of microinstabilities that are responsible for anomalous transport. TIM – trapped ion mode, TEM – trapped electron mode, ITG \((\eta_i)\) – ion temperature gradient mode, ETG \((\eta_e)\) – electron temperature gradient mode, CDBM – current diffusive ballooning mode, \(\delta p\) – skin-depth (mode); D – dissipative, C – collisionless, ES/EM – electrostatic/electromagnetic. [Source: *J. Plasma Fusion Res.* 76, 1280 (2000)]

all the terms on the \(LHS\), except \(-i\omega \rho_i\). This wave has the frequency \(\omega = \omega_{*e}\). It is marginally stable since \(\Im(\omega) = 0\), and it propagates on the flux surface in the electron diamagnetic direction. Since the magnetic curvature is neglected, \(\omega_{Di} = 0\), this mode does not rely on the toroidal geometry: it is a slab mode. It is not affected by electron collisions as long as the electron response is considered to be adiabatic. The first term of Eq. (2.21) coming from the compressibility-like \(v_\parallel \cdot \partial X\) term of Eq. (2.19) would give sound wave propagation along the field lines. Neglecting this term would mean that the ion inertia is assumed to be infinite, which is justified if the parallel phase velocity of the wave is much higher than the ion thermal velocity and the magnetic field is not strongly sheared. In the toroidal picture, this term describes the transit resonance, which might be neglected if the considered frequency range is much higher than the transit or bounce frequency of ions. FLR effects can be neglected when the wavelength is much longer than the
Larmor radius, \( k_\theta \rho_i \ll 1 \). Since \( \rho_e \ll \rho_i \), in the description of ion modes \( (k_\theta \rho_i \sim 1) \) the electron FLR effects can always be neglected.

In a local analysis of the ion response, taking the collisions into account by a simple energy-dependent Krook model \( C(g_i) = -\nu g_i / x^3 \), the ion gyrokinetic equation is an algebraic equation with the solution

\[
\begin{align*}
g_i &= \left[ \frac{e f_i}{T_i} \frac{\omega - \omega_{*i} \left[ 1 - \left( \frac{3}{2} - x^2 \right) \eta_i \right]}{\omega - k_\parallel v_\parallel - \omega_{DT} (x^2_\perp/2 + x^2_\parallel) + i \nu / x^3} J_0(z_i) \phi, \right. \\
&\quad \left. \omega - k_\parallel v_\parallel - \omega_{DT} (x^2_\perp/2 + x^2_\parallel) + i \nu / x^3 J_0(z_i) \phi, \quad (2.25) \right)
\end{align*}
\]

where \( x = v / v_{Ti} \) and \( \omega_{DT} = -k_\theta v^2_{Ti} / \Omega_i R \). We have introduced \( \eta_i = L_{ni} / L_{Ti} \), which is a crucial parameter in ITG theory [37], and we have taken \( \theta \to 0 \). The ion density response appearing in the dispersion relation is the velocity integral of this expression multiplied by \( J_0(z_i) \). The integral contains poles coming from the resonances in the denominator. These terms - the transit, the magnetic drift, and collisional resonances – can destabilize the mode. Usually the first of these resonances dominate and correspond to the slab and the toroidal ITG modes, respectively. We note that if all resonances and FLR effects are neglected (as in the previous example) the \( (3/2) \eta_i \) and \( x^2 \eta_i \) terms in the numerator cancel out in the velocity integration and the mode is not affected by temperature gradients.

In general, the collision term contains a differential operator in velocity space and the parallel ion dynamics term makes the problem a differential equation in real space. Moreover, considering the full dispersion relation with a bounce-averaged electron response term, one obtains an integro-differential equation in phase space, which would be intractable analytically without further approximations. One can make use, for example, of the following considerations: If \( \nu_{ei} / \omega \ll 1 \), then \( \nu_i / \omega \) is negligibly small. For \( \omega_{bi} \ll \omega_{DT} \) the parallel dynamics term is much smaller than the magnetic drift term \( k_\parallel v_\parallel \ll \omega_{DT} \). In addition, if the wavelength is comparable to or longer than the ion gyro-radius, the electron finite Larmor radius can be neglected \( z_i \ll 1 \Rightarrow z_e \sim 0 \). Furthermore, one can try to identify the terms that shape the mode structure, and others setting the mode frequency. Additionally, if \( \omega \ll \omega_{be} \) the parallel dynamics dominates the circulating electron response; these electrons can almost freely follow the potential perturbation and therefore have a Boltzmann response. If the drift frequency for thermal velocities, \( \omega_{DT} \), is much lower than the mode frequency, the drift resonance is expected to play minor role. Then the expansion of the integrand in the smallness of \( \omega_{DT} / \omega \) – the so called ”non-resonant expansion” – might be useful,
but the validity of this approximation turns out to be limited [41].

2.4.1 Ion temperature gradient mode

The most important microinstability affecting the ion thermal confinement is the ion temperature gradient mode (ITG or $\eta_i$-mode), which is a passing particle mode in the $k_\parallel v_{Ti} \lesssim \omega \ll k_\parallel v_{Te}$ frequency range. Classically the mode was investigated assuming adiabatic electron response and the ion response was calculated as (2.25). Depending on whether the mode is destabilized by the $k_\parallel v_\parallel$ or the $\omega_D$ resonance the mode is called slab- or toroidal ITG. The former mode, which is basically a coupled drift wave/ion acoustic wave in the presence of a radial ion pressure gradient, appears even if we neglect the magnetic curvature. Its typical frequency can be estimated as $\omega \sim \left( k_\parallel^2 v_{Ti}^2 \omega_e \eta_i \right)^{1/3}$ and it propagates in the ion diamagnetic direction [37].

![Figure 2.3: Ion heat diffusivity as a function of the logarithmic temperature gradient for different gyro-kinetic and gyro-fluid codes. [Source: Phys Plasmas 7, 969 (2000)]](image)

In toroidal geometry the curvature replaces the parallel ion compressibility as the main driving mechanism. The quasi-neutrality condition provides an eigenfunction problem in the ballooning angle with solutions peaking near $\theta = 0$, i.e., in the bad-curvature region, thus showing a "ballooning structure". In this region, the magnetic drift acts to destabilize the mode through the ion temperature gradient [37].
Chapter 2. Turbulent transport and microinstabilities

The transit resonance term and the FLR effects also play an important role in shaping the ballooning eigenfunction $\phi(\theta)$. This mode also propagates in the ion diamagnetic direction and has a frequency approximately $\omega \sim (\omega_{ni}\omega_D\eta_i)^{1/2}$.

An important feature of the ITG mode which appears in both slab and toroidal cases is that the mode is stable below a critical temperature gradient or $\eta_i$ threshold, as illustrated in Fig. 2.3.

Within the adiabatic electron approximation and for pure hydrogen plasma, the mode has no unstable roots for $\eta_i = 0$. Finite $k_\parallel$ is found to be stabilizing through Landau damping, and which thus introduces a $q$ dependence, since for ballooning type modes in tokamaks the parallel wavelength must be comparable to the connection length $\sim Rq$ [42]. Introducing a non-adiabatic trapped electron response, a new root, the trapped electron mode appears which can be clearly distinguished from the ITG mode if $\eta_i, \eta_e$ and the trapped electron fraction are high, although there are parameter regimes where the two modes form a single hybrid mode [43], which can be unstable for $\eta_i$ values lower than the classical threshold value (unstable modes with $\eta_i = 0$ are possible in case of impure plasma as well). With a finite non-adiabatic electron response the stability boundaries are modified, mainly for higher values of the logarithmic density gradient.

We note that the slab mode can be relevant in toroidal geometry as well, when the magnetic drift frequency becomes much smaller than the diamagnetic frequency (when $\varepsilon_n$ is small), which is typical in the edge region, where the density profile is not flat [8].

Sheared $E_r \times B$ flows, so called zonal flows, driven by self-generated radial electric fields $E_r$, have a strong linear stabilizing effect [44]. In addition, the radial correlation length of turbulent structures is decreased by the flows, which leads to reduced transport for a given fluctuation level. This double effect is shown in Fig. 2.3 where the ion heat diffusivity curve showing higher threshold corresponds to higher $E_r \times B$ velocity. This kind of turbulence suppression is thought to be important for the non-linear self regulation of the plasma and transport barrier formation.

2.4.2 Trapped electron mode

The magnetic field strength in tokamaks decreases from the inboard side of the torus ("high field side" or HFS) towards the outboard side ("low field side" or LFS) approximately as $1/R$. Thus the particles — behaving like small magnetic dipoles following the field lines — experience
a magnetic mirror force. An $\mathcal{O}(\sqrt{\epsilon})$ fraction of particles with parallel velocity at the outboard midplane lower than $v_\perp \sqrt{B_{\text{max}}/B_{\text{min}}} - 1$ is reflected back from the high-field region, bouncing back and forth. These trapped particles spend most of their time in the bad-curvature region, the LFS, thus the curvature drift has a preferred direction (while this effect averages out for the circulating particles), and the associated local electrostatic fields drive $\mathbf{E} \times \mathbf{B}$ drifts giving rise to micro-scale instabilities. The trapped particles cannot follow the electrostatic perturbations freely even if their inertia is negligibly small (mathematically: the $k_{\parallel} v_{\parallel}$ term vanishes on average over the trapped orbits), therefore they behave non-adiabatically.

If collisions are not too frequent to de-trap the trapped particles under a bounce period, various kinds of trapped particle instabilities can arise. On the other hand, the collisional de-trapping can turn non-adiabatic particles to adiabatic [45]. Therefore collisions play a crucial role in the theory of trapped particle instabilities. The collision frequency $\nu$ is usually defined by the frequency of $\pi/2$ angle scattering. It is however convenient to define a higher, effective collision frequency $\nu_{\text{eff}} = \nu/\epsilon$ describing the frequency collisional detrapping, which requires a smaller amount of scattering. Since the TEM requires low electron-ion collision frequency, ion-ion and ion-electron collisions can be neglected.

The trapped electron mode is one of the most important microinstabilities which can dominate the transport in the presence of internal transport barriers and certainly significantly contributes to the anomalous fluxes in tokamaks [36]. The dissipative TEM is destabilized by the combined effect of the electron temperature gradient and the collisional de-trapping of the non-adiabatic trapped electrons. The response of the circulating particles is dominated by the parallel dynamics due to the small electron inertia, therefore the neglect of the $\mathcal{O}(\frac{\omega}{k_{\parallel} v_{T_e}}) \ll 1$ non-adiabatic circulating electron response is adequate [14]. For even lower collisionalities the electron magnetic drift frequency resonance destabilize the mode, which is then called the collisionless TEM.

It was found that modelling the collisions by a Krook operator causes a non-physical discontinuity at the trapped particle boundary. In reality collisions cause diffusion in velocity space, which is in contradiction with such discontinuous solutions. In the low collisionality regime – in the sense that $\nu_{\text{eff},e}/\omega \ll 1$ – a boundary layer appears at the trapped-passing boundary which continuously joins the regions where the non-adiabatic response is finite and where it is almost zero. The contribution
of this layer in the trapped electron response is proportional to \( \sqrt{\nu} \) [46].

The main drive of the TEM is the electron temperature gradient; in this aspect is similar to the ETG mode, although TEM wavelengths are typically much larger than the electron gyro-radius. The maximum growth rate occurs at \((k \theta \rho_i)^2 \lesssim 1\). For high collisionalities the growth rate \( \gamma \) varies approximately as \(1/\nu\). The TEM stability boundary shows a strong dependence on collisionality and the FLR parameter [14].

### 2.5 Role of collisions in transport

In the absence of microinstabilities the particle and heat fluxes across the magnetic surfaces are determined by collisional transport processes. In the presence of collisions, the particles do not always follow orbits determined by only macroscopic fields, but their trajectory can be changed in a random walk fashion due to Coulomb collisions. In *classical* collisional transport the dominant process is the random walk between Larmor orbits. In toroidal geometry the width of trapped ”banana” orbits plays the role of the step length and the characteristic time of particle detrapping becomes the step time. Although this so-called ”neoclassical” transport is much larger than its classical counterpart, it is nevertheless considerably smaller than the turbulent transport, except in transport barrier regions where they can be comparable. Collisions play an important role in the turbulent transport as well, since collisional dissipation affects the non-adiabatic response of the particles.

In gases collisions change the velocity of an atom almost instantaneously, so that its trajectory in phase space is a continuous set of line segments (on the appropriate timescale). In plasmas, on the other hand, each particle is in a continuously running Coulomb interaction with a large number of other particles being closer than a few Debye lengths (from those inside of the Debye sphere of the particle) [47]. Since small-angle scattering thus dominates, the phase-space trajectory of a single particle is a smooth curve, and the collision operator – describing the variation of the distribution function due to collisions – can be expressed as the \( \Delta t \to 0 \) limit of

\[
C \left[ f(\mathbf{v}, t) \right] = \partial_{v_i} \left( \frac{\langle \Delta v_i \rangle}{\Delta t} f \right) + \partial^2_{v_i v_j} \left( \frac{\langle \Delta v_i \Delta v_j \rangle}{2\Delta t} f \right) - \ldots, \tag{2.26}
\]

where the expectation value is denoted by angle bracket. The first term in Eq. (2.26) is responsible for the collisional drag on the particle while the second term describes diffusion in velocity space.
The $\langle \Delta v_i \rangle$ and $\langle \Delta v_i \Delta v_j \rangle$ quantities can be calculated by a statistical description of binary Coulomb collisions between plasma particles. The $\int dr/r$-like integral for the possible impact parameters do not need to be evaluated over an infinite range, since the particles being outside the Debye sphere do not contribute to the integral and the smallest distance between the colliding particles $r_{\text{min}}$ is also finite. So the integration limits are to be cut off at $\lambda_D$ and $r_{\text{min}}$, which leads to the appearance of the Coulomb logarithm parameter $\ln \Lambda = \ln(\lambda_D/r_{\text{min}})$ in the collision frequency, which is closely related to the number of particles inside the Debye sphere and being typically in the range $15 - 20$ for the cases of interest.

A particle is continuously in interaction with several other particles, and the small-angle scattering "events" strongly dominate, so the quantities, such as mean-free path or collision time from the classical picture of collisions are to be reconsidered in this context. The collision time $\tau$ is defined as the time which is required for a significant angle (usually $\pi/2$) change in the direction of the particle velocity as a result of the cumulated effect of Coulomb interactions. The collision frequency is defined as $\nu = 1/\tau$. The electron-ion collision frequency depends on electron mass $m_e$, electron temperature $T_e$, ion density $n_i$ and ion charge $Z$ in the following manner \[47\]

$$\nu_{ei} \propto \frac{e^4 n_i Z^2 \ln \Lambda}{\epsilon_0^2 \sqrt{m_e^2 T_e^3}}.$$ (2.27)

Since $\nu_{ei}$ is independent of $m_i$, the total electron-ion collision frequency in the presence of several ion species can be written as $Z_{\text{eff}} \nu_{ei}(Z \to 1)$, with the effective ion charge $Z_{\text{eff}} = (\sum_j n_j Z_j^2)/n_e$, where the sum runs over the different ion species. Due to the high ion-to-electron mass ratio much higher number of elementary interactions is needed for an ion to significantly change its velocity as a result of collision by electrons than vice versa. Accordingly, the relative magnitudes of the different collision frequencies are $\nu_{ei} = \nu_{ii} \sqrt{m_i/m_e} = \nu_{ie} m_i/m_e$.

Making use of Eq. (2.26), the most general collision operator, the so-called Fokker-Planck operator \[30\], describing the collision of two species characterized by their distribution functions $f_a$ and $f_b$ can be derived to
Chapter 2. Turbulent transport and microinstabilities

be \cite{48}

\[
C_{ab} \left[ f_a(v), f_b(v') \right] = -\frac{e_a^2 e_b^2 \ln \Lambda}{8\pi e_0^2 m_a^2} \nonumber
\]

\[
\times \partial_{v_k} \int \frac{u^2 \delta_{kl} - u_k u_l}{u^3} \left( \frac{f_a(v)}{m_b} \partial_{v'_l} f_b(v') - \frac{f_b(v')}{m_a} \partial_{v_l} f_b(v) \right) d^3 v',
\]

where \( u = v - v' \) is the relative velocity of the colliding particles. In several cases it is useful to derive an approximate, model collision operator that is simpler, and therefore less accurate than the one given above, but still models the physical phenomena that are important for the problem. It is always required that the collision operator drives the system towards local thermodynamic equilibrium. In particular, if \( f_a \) and \( f_b \) are two Maxwellian distributions with equal temperatures and mean velocities the operator should vanish.

Since electrons are much lighter than ions, the dominant process in the electron-ion collisions is pitch-angle scattering, which is velocity diffusion on a constant-energy surface. This process tends to make the electron distribution isotropic in the ion rest frame. The energy transfer is small due to the high ion-to-electron mass ratio \( m_i/m_e \), so the electron speed is approximately conserved if the ions are stationary \( (V_i = 0) \). Furthermore, if the gyro-angle dependence of the collisions can be neglected the collision operator reduces to the pitch-angle scattering operator

\[
C_{ei} \approx \nu_{ei} \frac{\mathcal{L}}{x_e^3} \equiv \nu_{ei} \frac{1}{2x_e^3} \partial_\xi \left( 1 - \xi^2 \right) \partial_\xi,
\]

where \( \xi = \cos \theta \) with the pitch-angle \( \theta \).

If one wants to investigate the collisional de-trapping of trapped electrons, it is sometimes enough to keep only a pitch-angle scattering model operator of electron-ion collisions since trapping depends only on the cosine of the pitch angle, \( \xi = v_\parallel/v \). In other cases it is merely sufficient that the collision operator drives the particle distribution towards a Maxwellian, so that it reduces the perturbed part of the distribution. An example of such an operator is the so-called \textit{Krook model}, \( C = -\nu \hat{f} \), where an energy dependence can be included in the collision frequency \( \nu \). For other applications it can be important that the treatment of collisions is momentum conserving or that it incorporates other physical effects, such as parallel velocity diffusion or deceleration of the colliding species. A systematic derivation of model collision operators and a good summary of previous works is given by Hirshman and Sigmar \cite{49}.

22
Besides that collisions drive the classical and neoclassical transport, it was found both experimentally and in simulations that they can strongly affect anomalous particle transport [50]. Electron density profiles in tokamak cores are usually not completely flat, although the particle sources are mainly localized in the periphery of the plasma. Neoclassical theory predicts an inward particle transport, due to the Ware pinch [51]. However this effect is not sufficiently strong to explain the density peaking found experimentally, which implies the existence of an anomalous component of the pinch. Experimentally, density peaking is found to increase with decreasing collisionality [50,52–54], as shown in Fig. 2.4. This phenomenon is crucial for reactor relevant fusion experiments operating with high temperatures and thus having low collisionalities.

If collisions are neglected, the inward particle fluxes can be explained by gyrokinetic theory [55]. The transport is dominated by ITG driven turbulence in most cases, and ITG modes produce an inward particle flux through magnetic curvature effects and thermodiffusion, as predicted by both linear theory and nonlinear gyrokinetic simulations [41]. However, the particle flux reverses when collisions are introduced, and the sign change occurs at much lower collisionalities than achieved in present fusion experiments [56]; the theory seems to be in contradiction with the experimental results. In fact, the particle flux shows a very strong collisionality dependence as illustrated in Fig. 2.5; showing the result of a nonlinear gyrokinetic simulation using a pitch-angle scattering collision.
operator. We note that gyro-fluid simulations predict somewhat higher collisionality for the reversal of particle flows \[56\]. It has been reported recently in Ref. \[57\], that the high wave number part of the turbulent spectrum contributes to an inward particle flow which might be able to cancel the outward flux predicted at wave numbers where the particle flux spectrum reaches its maximum \((k_{\perp} \rho_i \sim 0.1 - 0.2, \text{at the maximum of } \gamma/k_{\perp}^2)\). However, the discrepancy between theory and experiment is not completely resolved yet.

The collisionality dependence of the particle flux at low electron-ion collision frequencies suggests that the non-adiabatic electron response is affected (note that \(\Gamma \propto \Im(\hat{n}_e) \text{ and } \nu_i \approx 0\)). The circulating electrons are expected to have a quite weak non-adiabatic response which is almost independent of collisionality \[41\]. This conclusion is supported by Fig. 2.5, where the trapped and passing contributions in the particle flux are also indicated. The \(\sqrt{\nu_{ei}}\)-like collisionality dependence can be interpreted with the development of a boundary-layer at the trapped-passing boundary in the \(\nu_{ei} \ll 1\) limit.

We investigate the collisionality dependence of the quasilinear particle flux for weakly collisional plasmas by means of a WKB solution of the electron gyrokinetic equation, where the electron-ion collisions are modelled by the pitch-angle scattering operator. The solution reproduces the mentioned boundary layer development and is valid for collision frequencies up to \(\nu/(\epsilon \omega) \lesssim 1\). In paper A, the effect of collisions on the mode frequency is neglected (which is found to be justified for ITG modes) and the non-resonant expansion in magnetic drifts is applied. In pa-
per B we extend the validity of our model by taking collisional effects into account through the dispersion relation as well, and calculating the density responses without assuming the magnetic drift frequency to be small. Furthermore, we point out that, while the ITG mode frequency shows only a weak dependence on collisionality far from marginal stability, at high enough values of $a/L_n$ the ITG mode is strongly stabilized by collisions. Finally, in paper B we also investigate collisional effects on the ion and electron quasilinear energy fluxes $q_i$ and $q_e$, and we find that they are only weakly dependent on collisionality, in agreement with experimental results (in L-mode [58]).
Chapter 3

Beam emission spectroscopy

The advance in the understanding of anomalous transport, such as the development of plasma turbulence models, strongly rely on experimental data measured by various kinds of plasma diagnostic tools. Since in such high temperature systems, the methods based on physical contact of the measuring device and the plasma are quite limited, the diagnostics either passively collect radiation or particles emitted by the plasma (passive diagnostics), or observe the interaction of the plasma with some material/radiation introduced externally (active diagnostics) [59]. Beam emission spectroscopy (BES) is an important, widely used active diagnostic tool of fusion plasmas, which is based on the observation of light emitted by a high energy neutral beam injected into the plasma [60]. The measured intensity distribution – corresponding to the spontaneous emission from the highest population excited atomic state, – the light profile, provides information on the distribution of plasma parameters affecting the beam evolution.

Heating beams (neutral beam injection, NBI) – observed tangentially to the magnetic field lines – are also used for beam emission spectroscopy [61,62], although in the thesis we focus on (accelerated) beams used only for diagnostic purposes [63–66]. The diagnostic beams have much lower beam current (∼mA) than the NBI beams, and due to the attenuation of the beam in the plasma, mainly the outer plasma regions can be probed by them. Since the density of beam atoms is ∼10^{14} \text{ m}^{-3} (which is low compared to ∼10^{19}−10^{20} \text{ m}^{-3} plasma densities), the momentum transfer from the beam is negligibly small, and the quantity of
deposited beam material is usually too low to noticeably modify the $Z_{eff}$ in the plasma, the method is considered to be non-intrusive [67]. Further advantage of BES that it is not line integrated, but a well localized measurement of plasma parameters.

The electron density, the temperature of plasma particle species (mainly $T_e$) and the distribution of impurities are the relevant parameters determining the beam evolution [68]. Due to the relatively weak temperature dependence of the reaction rates of alkali elements, they are well suited for electron density measurements. We restrict our studies to the prevailing alkali beam emission spectroscopy (noting that helium beams are also used for simultaneous temperature/density measurements, where the different energy dependence of cross sections for singlet and triplet states is utilized [69,70]).

One of the main purposes of alkali BES is the electron density fluctuation measurement [63,65] (with $\sim 0.5 \mu s$ temporal resolution [23]) which provides useful – statistical or time-resolved – information on turbulence, such as frequency-, wavenumber spectra, flow velocities, spatial (1-D or 2-D) and temporal correlations, or even snapshots of the turbulent structures. In fluctuation measurements linear – but not local – response in emitted intensity to density perturbations is assumed, which is dependent on the equilibrium (“static”) density profile. The time resolution of such measurements is limited by the photon statistics which depends on the achievable beam current and the efficiency of the observation. Results of BES fluctuation measurements are shown in Fig. 3.1, where – in the left figure – the broadband turbulent spectrum and the dramatical reduction of the density fluctuations in an L-H transition is plotted, while the frequency spectrum of density fluctuations in different radial positions is plotted in the right figure (the shift of the spectrum maximum due to the higher ion Larmor radius in the higher temperature regions is clearly visible).

An other important application of alkali BES is the electron density profile measurement [60,64,68]. This measurement, relying on the knowledge of the derivative of the light profile, requires smooth, time averaged light profiles, which sets its time resolution. The highest achieved time resolution for independent profile measurement published is $\gtrsim 50 \mu s$, although this is purely a technological limit, since with higher beam current and observation efficiency, and accordingly, better photon statistics higher time resolutions could be achieved. The spatial resolution of both measurement modes is limited by the characteristic distance
Figure 3.1: Deuterium BES measurement on the DIII-D tokamak. Left: spectrogram showing the evolution of turbulent density fluctuations in an L-H transition ($r/a = 0.65$). Right: Power spectra of fluctuations in different radial positions. [Source: Plasma and Fusion Research 2, S1025 (2007)]

covered by a beam atom under the spontaneous decay of the observed transition ($\gtrsim 5$ mm [23]). The evolution of the edge-SOL density profile measured by Li-BES diagnostics is shown in Fig. 3.2 during an ELM event.

Figure 3.2: Lithium BES density profile measurement on the ASDEX-Upgrade tokamak. Time resolution: 50 $\mu$s. The curves show the time evolution of electron density at different radial positions ($\rho_{pol} = 1$ is the last closed flux surface). An ELM (Type-I) appears at $\approx 2.046$ s. [Source: Plasma Phys. Control. Fusion 50, 085009 (2008)]

In NBI BES measurements the beam (which is usually observed
tangentially to the magnetic field lines) is wide enough to make 2-dimensional turbulence measurements possible for a single beam position [71]. By means of scanning the beam, poloidally resolved measurements are also feasible with the $\sim 1 - 2$ cm width diagnostic beams [63]. In these measurements, the scanning frequency is higher than the achievable frequency resolution of the profile measurements. In several cases the beam is observed from the same poloidal plane, and a time averaged signal of the fluctuation measurement is used for calculating the density profile which is, in turn, used as an input to the evaluation of the fluctuation measurements. In this configuration, the density calculation is based on the light profile from a beam which is several times wider than the physical beam width. Although, in many experiments a one-dimensional beam is considered in the density calculations. In paper C, we point out that even in the 1-D measurements, the result might be significantly affected by the finite beam width.

An important indirect application of BES is the charge exchange recombination spectroscopy (CXRS) [72], where charge exchange processes between beam atoms and impurities are used to induce characteristic emission of the impurities, making the impurity measurements spatially localized. This diagnostic method provides information on the concentration, temperature and flow velocity of different impurity species. In a subtle application of BES, the motional stark effect (MSE) diagnostic [73], the pitch angle of magnetic field lines, therefore plasma current density can be locally measured, utilizing the splitting of atomic spectral lines to orthogonally polarized components due to the motional electric field that is present in the rest frame of the beam atoms.

In a typical alkali BES measurement set-up [63] the first component is the ion gun, where ions of the beam material are extracted from a ceramic emitter and then in the ion optics they are accelerated in a $10 - 100$ kV electrostatic field to form a collimated $1 - 10$ mA ion beam. In 2-D measurements the beam deflection is done by an alternating perpendicular electrostatic field before the beam enters into the neutralizer chamber which usually contains alkali metal vapor. The efficiency of the neutralization by charge exchange processes is $70 - 80\%$, and the still ionized fraction of the beam is to be removed by a magnetic field, ending on an ion dump. The beam ends in the plasma and is being observed by the detecting system. The efficiency of the system is determined by the dimensions of the collecting optical element, the losses on the whole optical system, and the quantum efficiency of the detector collecting the
light filtered by a monochromator to the Doppler shifted frequency of
the observed spectral line.

The evolution of the beam in the plasma is accurately described by
the collisional radiative model [72] which considers the collisional and
spontaneous transition processes of the electron configuration. Each
collisional process is characterized by a rate coefficient \( \left[ \text{m}^3\text{s}^{-1} \right] \) defined
as

\[
R = \int d^3v \sigma (|\mathbf{v} - \mathbf{v}_B|) |\mathbf{v} - \mathbf{v}_B| f_\alpha (\mathbf{v}),
\]

where \( \mathbf{v}_B \) is the velocity of the beam atoms, \( \sigma \) is the cross section of
the process and \( f_\alpha \) is the velocity distribution of the colliding species,
considered to be a Maxwellian. The rate coefficients depend paramet-
rically on the beam energy and the temperature \( T_\alpha \). In terms of the
rate coefficients, the evolution of the occupation densities of the atomic
states is described by the rate equations, which read in the rest frame of
beam atoms

\[
d_t N_k = -N_k \sum_{j(<k)} A^{jk} + \sum_{j(>k)} A^{kj} N_j
+ \sum\limits_\alpha n_\alpha \left\{ -N_k \left[ R_\alpha^{+k} + \sum_{j(\neq k)} R_\alpha^{jk} \right] + \sum_{j(\neq k)} R_\alpha^{kj} N_j \right\},
\]

where the different plasma species are indexed by \( \alpha \), and the \( j \) and \( k \)
indices run over the registered atomic levels; \( n \) denotes particle density,
\( N \) is the population of an atomic state. \( A^{kj} \) is the spontaneous transition
frequency and \( R_\alpha^{kj} \) is the rate coefficient corresponding to the \( j \to k \)
transition. The ionized state is denoted by \( + \), and the ionized beam atom
is considered to be lost from the beam. In a simulation the evolution of
a finite number of levels are followed, and the populations of the higher
principal quantum number states are neglected.

As an approximation, the effect of impurity collisions can be taken
into account through one representative impurity with the average
impurity charge \( q = \sum_{\alpha'} n_{\alpha'} Z_{\alpha'}/\sum_{\alpha'} n_{\alpha'} \), where \( \alpha' \) runs over the impurity
species. From the quasineutrality condition the density of this impu-
rity is \( n_q = n_e C \) and the bulk ion density is \( n_i = n_e (1 - qC) \) with
\( C = (Z_{\text{eff}} - 1)/(q(q - 1)). \) It is convenient to transform the rate
equations into the laboratory frame, and write them in a more compact
form [68]

\[
d_x N_k = \sum_j \left[ n_e(x)\tilde{a}^{kj}(x) + b^{kj} \right] N_j(x) \quad (j, k = 1, \ldots, m),
\]
where \( m \) is the number of the registered atomic states, the reduced rate coefficient matrix \( \tilde{a} \) depends on the position \( x \) through its \( T_\alpha, q \) and \( Z_{\text{eff}} \) dependence and it is defined by 
\[
\tilde{a}_{kj} = \left( R_{ej}^{kj} + (1-qC)R_{ij}^{kj} + CR_{qj}^{kj} \right)/v_B
\]
for different indices and 
\[
\tilde{a}_{jj} = -\left( R_{ej}^{jj} + (1-qC)R_{ij}^{jj} + CR_{qj}^{jj} \right)/v_B - \sum_{k \neq j} \tilde{a}_{kj},
\]
and the spontaneous transitions are taken into account through the constant matrix 
\[
b_{kj} = A_{kj}/v_B, \quad b_{jj} = -\sum_{k \neq j} A_{kj}/v_B.
\]
Since the characteristic time spent by the beam atoms in the plasma before their ionization \((\sim 0.1 \mu s)\) is much shorter than the time resolution of the density profile measurements, the time evolution of plasma parameters is neglected during this time. However, correction for this effect might be important in fluctuation measurements.

The emitted light intensity is proportional to \( N_\iota J_A \phi_\iota/v_B \), where the \( J \) is the current density of the beam and the \( \iota \rightarrow \phi \) transition is observed. Since in alkali BES measurements one spectral line is observed, the evolution of only one atomic population \( N_\iota \) is known directly. There exists a point, where the collisional processes acting to populate and de-populate the \( \iota \) level equalize, therefore the evolution of this population becomes independent of electron density, and the measurement is insensitive in the vicinity of this point. The existence of this singular point makes the measurement evaluation a quite complicated and stiff numerical problem [64].

### 3.1 Turbulence measurements

Due to the turbulence which is always present in fusion plasmas, there is an incessant fluctuation in the distribution of the plasma particle species. Spatial and temporal correlations, frequency and wave number power spectra of the density fluctuations can be measured by means of alkali BES, mainly in the long wavelength region \((k_\theta \rho_i \ll 1)\). In addition, the direct time-resolved 2-D turbulence imaging measurements have already been demonstrated [71].

In turbulence measurements it is assumed that the electron density distribution along the beam can be decomposed to a slowly varying density profile and a fluctuating density \( n_e(x,t) = n_{e0}(x) + \hat{n}_e(x,t) \) [74]. The time average of \( \hat{n}_e(x,t) \) vanishes, and on the time scale of fluctuations the profile is considered to be static. Here, for simplicity, we consider one dimensional fluctuation measurement and neglect the finite thickness of the beam, although the quantities to be introduced can be generalized to two dimensions.
3.1. Turbulence measurements

The measured light profile $S$ can be decomposed accordingly to a static and a fluctuating part $S(x,t) = S[n_{00}(x), T_{00}(x)] + \hat{S}(n_{0}, T_{0})$, where the static part is a complicated nonlocal function $S$ of the equilibrium parameter profiles, which can be determined by solving the rate equations (3.3), and the fluctuation part depends on both the static and fluctuating parts of plasma parameters. However, for alkali beams, the effect of temperature and impurity content fluctuations on $\hat{S}$ can be neglected. Then, the fluctuating part of the measured light profile can be written in terms of the density fluctuation transfer function $h(x, x')$ as

$$\hat{S}(x, t) = \int_{0}^{x} \hat{n}_{e}(x', t) h(x, x') dx',$$  

(3.4)

where the transfer function considered to be dependent only on the static part of the plasma parameter profiles. (In fact, the value of $\hat{S}(x, t)$ depends on the density fluctuations at the retarded time $t - (x - x')/v_{B}$, but in the present reasoning, this effect is neglected.) We note, that the assumption of linear light response to density fluctuations breaks down if the amplitude of the density fluctuations is not small enough compared to the background density.

Often, in high frequency fluctuation measurements - due to the low number of detected photons - the signal-to-noise ratio is close to one, which makes the time-resolved evaluation of the measurement impossible. However, statistical information on turbulence can be obtained from the cross-correlation function of the density fluctuations, which is defined as

$$C_{n}(x_{1}, x_{2}, \tau) = \frac{1}{T} \int_{0}^{T} \hat{n}_{e}(x_{1}, t) \hat{n}_{e}(x_{2}, t + \tau) dt,$$

(3.5)

where $T$ is the length of the time window. From the measured light profile fluctuations, the cross-correlation function of the light profile fluctuations can be obtained, that is defined similarly

$$C_{S}(x_{1}, x_{2}, \tau) = \frac{1}{T} \int_{0}^{T} \hat{S}(x_{1}, t) \hat{S}(x_{2}, t + \tau) dt.$$  

(3.6)

This function can be determined quite accurately by statistical averaging. It can be shown using Eq. (3.4) that these correlation functions satisfy the following relationship

$$C_{S}(x_{1}, x_{2}, \tau) = \int_{0}^{x_{1}} \int_{0}^{x_{2}} h(x_{1}, x') h(x_{2}, x'') C_{n}(x', x'', \tau) dx' dx''.$$  

(3.7)
This integral equation has to be solved in order to determine $C^n$ from the measured $C^S$, which can be done by, e.g., using regularization methods [74], if the fluctuation transfer function $h(x, x')$ is known. This, in turn, can be calculated recognizing that $h(x, x_0) = \int_0^x \delta(x' - x_0) h(x, x') dx'$. The transfer function can be expressed as

$$h(x, x_0) = S [n_e + \delta(x' - x_0)] - S [n_e],$$

in other words, we superimpose an elementary fluctuation, a Dirac delta, at point $x_0$ on the static density profile, and monitor the difference appearing in the light profile. Clearly, Eq. (3.8) implies that the fluctuation measurement cannot be evaluated without the knowledge of the static density profile.

![Figure 3.3: Density fluctuation transfer function (output of the RENATE simulation code; simulated Li-BES measurement for a high density discharge on the COMPASS tokamak). The darker regions correspond to higher positive response in the measured light profile. The zero response (such as for $x' > x$) and the negative response are indicated with contours.](image)

As an example, in Fig. 3.3 a density fluctuation transfer function, $h(x, x')$, is contour plotted (Li-BES, the input profiles are taken from the COMPASS #30866 H-mode discharge at 156 ms). For low values of $x'$ there is a positive response due to the extra collisional population of the initial state of the observed transition ($2p$ for Li). The response decreasing exponentially with $x$ as this atomic state spontaneously decays. A bit farther in $x$ from $x = x'$ the response becomes negative, since the collisions contribute to the ionization of beam atoms as well. For higher values of $x'$ the positive response gets smaller, and at a certain point (in this case around 5.5 cm) the response vanishes at
3.2 Electron density measurements

Good spatial and temporal resolution electron density profile measurements [23, 64, 68] in the outer regions of fusion plasmas are of great importance, providing useful information, e.g., on the edge transport barrier formation, ELM activity, and, as we pointed out, the knowledge of the density profile is also an important input of BES fluctuation measurement.

In the direct problem – such as in a BES measurement design – we are interested in the measured light profile given the plasma parameter profiles along the beam line. It consists of the simulation of beam evolution and the modeling of the observation of the emitted light. Considering an ideal (one dimensional) beam, the beam evolution can be calculated by the numerical integration of the rate equation system (3.3). The initial condition is given by the assumption that all the beam atoms are in ground state at the point, where the beam enters the plasma \( N_j(x = 0) = \delta_{1j} \) (here the ground state is labeled by the index 1) [64].

![Graph of atomic populations along beam line](image)

**Figure 3.4:** Evolution of atomic populations along the beam line (normalized to \( N_1(0) \)). Li-BES simulation for the COMPASS tokamak. The light profile is proportional to the \( 2p \) population.

The evolution of the atomic populations as a function of distance along the beam \( x \) is plotted in Fig. 3.4 for the same plasma parameter
profiles as Fig. 3.3. Initially only the ground-state (2s) is populated, then the collisional and spontaneous processes drive the levels towards a secular equilibrium with the ground state, and in the a final phase collisional ionization is the dominating process and the beam attenuates.

In a measurement evaluation the inverse problem is solved, where the electron density is to be calculated from the measured light profile \( I(x) \) and the profiles of electron temperature, \( Z_{\text{eff}} \) and average impurity charge along the beam line. The classical solution of the inverse problem starts from the rate equation for the initial state \( \iota \) of the observed transition, that is

\[
d_x N_{\iota} = \sum_j \left[ n_e(x) \tilde{a}^{\iota j}(x) + b^{\iota j} \right] N_j(x).
\] (3.9)

Using that the light profile \( I(x) \) is proportional to \( N_{\iota} \), from Eq. (3.9) the electron density can be expressed as

\[
n_e(x) = \frac{d_x (\ln I) - \sum_j \frac{N_{\iota} N_j b^{\iota j}}{N_{\iota} \tilde{a}^{\iota j}}}{\sum_j \frac{N_{\iota} \tilde{a}^{\iota j}}{N_{\iota} \tilde{a}^{\iota j}}}.
\] (3.10)

Since the relative populations are not known \textit{a priori}, Eq. (3.10) has to be solved simultaneously with the direct problem [64]. Note that this method does not require the absolute value of the \( N_{\iota} \) population, only an arbitrarily normalized light profile.

For experimentally relevant plasma densities, there exists a point along the beam (usually a bit after the light profile maximum, as it can be seen from Figs. 3.3 and 3.4) where the collisional population and de-population of level \( \iota \) equalize. Mathematically it means that Eq. (3.10) becomes of the form \( 0/0 \), therefore the stepwise simultaneous solution of Eqs. (3.10) and (3.3) terminates.

The calculations are usually stopped when the denominator falls below a certain value. Then the classical method is replaced by a technique, where the density is calculated as a fraction of integral quantities that are non-vanishing in the vicinity of the singular point [68]. This technique relies on the knowledge of the absolute \( N_{\iota} \) profile, and the parameter \( \alpha = N_{\iota}/I \) is found iteratively. Unfortunately, for self-consistent density estimation it is crucial that the whole emission profile is observed, which excludes low-density profile measurements. Furthermore the method is very sensitive to the accuracy of the \( \alpha \) parameter and easily becomes unstable.
3.2. Electron density measurements

Recently, a Bayesian probabilistic method has been developed [23], based on the solution of the direct problem. The measured data are compared to direct calculations, and the most probable density profile is chosen. This approach requires somewhat higher computational capacity, and the constraint of monotonicity of the $n_e$ profile, although it is much more stable than the previous techniques – even in the vicinity of the singular point – and allows for the evaluation of noisy data, accordingly, higher time resolution profile measurements. At the same time it provides the accuracy of the calculation in each point.

Nevertheless, none of these methods can give information on the density in the singular point since this point is not a numerical artifact of the methods, but – since in this point the evolution of the $N_e$ population becomes independent of collisions – no information of the density profile can be extracted from the light profile there. We also note that due to the properties of the inverse problem, light profiles that are only slightly different can give significant differences between the calculated density profiles.

In the density profile measurements it is crucial that the light profile is proportional to the $N_e$ profile (if the spatially varying efficiency of the observation is taken into account). Although, since the measurement of the light profile means integration along the lines of sight, this assumption is valid if the width of the beam is negligibly small compared to the scale length of the $N_e$ profile, since otherwise we integrate through beam parts that are in different stages of beam evolution. As we point out in paper C, the neglect of finite beam width might cause non-negligible error in the calculated density profile – regularly the underestimation of the pedestal density. By modeling the observation of a finite width beam the transfer function of the observation can be calculated which relates the measured light profile $S(x)$ to the emission density along the beam axis $I(x)$ (that is identical to the light profile of an ideal, infinitesimally fine beam). We developed a de-convolution based inversion algorithm, which – given the measured light profile – calculates $I(x)$, allowing for the use of the conventional density reconstruction methods considering one-dimensional beam.
Chapter 4

Summary

In the present thesis theoretical and experimental aspects of anomalous transport in tokamaks are addressed.

In the first part of the thesis, the most important electrostatic drift wave instabilities driving the turbulent transport, the ion temperature gradient (ITG) and trapped electron (TE) modes and the quasilinear fluxes driven by them are studied focusing on the effect of collisions.

In paper A, the collisionality dependence of quasilinear particle flux due to ITG and TEM modes is investigated analytically. For weakly collisional plasmas, we derive the WKB solution of the trapped electron gyrokinetic equation, where the collisions are modeled by the Lorentz operator. In this model the frequencies and growth rates are considered as input parameters, therefore the dependences on different parameters – such as collisionality – through the eigenfrequency are neglected, and we use a simple, purely real model ballooning potential.

In accordance with previously published gyrokinetic simulation results, we find that, far from marginal stability, the inward flux due to ITG modes – caused mainly by magnetic curvature effects and thermodiffusion – is reversed as electron collisions are introduced. However, if the plasma is close to marginal stability, collisions might even enhance the inward particle transport. We compare the results calculated by using the Lorentz operator and an energy dependent Krook operator and conclude that the form of the collision operator determines the scaling with collisionality and therefore affects the collisionality threshold where the particle flow reverses. The difference between the two models is larger close to marginal stability. We find that, for low collisionalities, due to the boundary layer development of the non-adiabatic electron
distribution function at the trapped-passing boundary, the collisional contribution in the particle flux is proportional to the square root of collisionalities.

In paper B, we improved our “COllisional Model of Electrostatic Turbulence” (COMET) regarding several aspects and focus on the stability of the ITG mode and the ITG-driven quasilinear fluxes. Here, also the ion response is calculated in the long wavelength limit, so that the mode frequencies are calculated from the quasineutrality constraint. We introduce a shear dependent imaginary part of the ballooning potential, which we motivated by a self-consistent variational solution of the ballooning eigenfunction problem. This is found to be important for the quantitatively accurate calculation of mode frequencies and fluxes. The improved model, where both the particle and energy fluxes are calculated, does not rely on the non-resonant expansion in magnetic drift frequencies.

We find that, although the frequencies and growth rates of ITG modes far from threshold are only weakly sensitive to collisionality, the temperature gradient threshold for stability is significantly affected by electron-ion collisions for high enough logarithmic density gradients. The decrease of collisionality destabilizes the ITG mode driving an inward particle flux, which leads to the steepening of the density profile, in agreement with the trend found in experiments on the collisionality dependence of density peaking. Closed analytical expressions for the electron and ion perturbed density and temperature responses have been derived; and simple, but quite accurate algebraic approximations for these quantities are given. The semi-analytical COMET model has been benchmarked to the recognized, state-of-the-art gyrokinetic code GYRO, and we found quantitatively good agreement – in the moderate shear regime – with linear GYRO simulations and qualitative agreement with nonlinear simulations.

In the second part of the thesis, the magnitude and characteristics of the error in alkali beam emission spectroscopy (BES) density profile measurements due to finite beam width are analyzed and a de-convolution based correction algorithm is introduced.

If the line of sight is far from tangential to the flux surfaces and the beam width is comparable to the scale length of the light profile, the observation might cause an undesired smoothing of the light profile, resulting in the underestimation of the measured light profile. In paper C, the characteristics and magnitude of this systematic error is studied;
a general estimation of the maximal relative error is presented depending on the electron density at the light profile maximum, beam width, $Z_{\text{eff}}$ and the relative directions of the lines of sight, the beam axis and the flux surfaces. We demonstrate a new de-convolution based correction method by its application in simulated BES measurements of the COMPASS and TEXTOR tokamaks. The method gives a good estimate of the emissivity along the beam line from the measured light profile so that the level of the remaining error in the calculated density after correction is of the order of the accuracy of the density profile calculation algorithm. The method allows the use of the conventional one dimensional density calculation algorithms even for configurations, where the finite beam width is not negligible.
References


REFERENCES


Included papers A-C
Collisionality dependence of the quasilinear particle flux due to microinstabilities

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The collisionality dependence of the quasilinear particle flux due to the ion temperature gradient (ITG) and trapped electron mode (TEM) instabilities is studied by including electron collisions modeled by a pitch-angle scattering collision operator in the gyrokinetic equation. The inward transport due to ITG modes is caused mainly by magnetic curvature and thermodiffusion and can be reversed as electron collisions are introduced, if the plasma is far from marginal stability. However, if the plasma is close to marginal stability, collisions may even enhance the inward transport. The sign and the magnitude of the transport are sensitive to the form of the collision operator, to the magnetic drift normalized to the real frequency of the mode, and to the density and temperature scale lengths. These analytical results are in agreement with previously published gyrokinetic simulations. Unlike the ITG-driven flux, the TEM-driven flux is expected to be outwards for conditions far from marginal stability and inwards otherwise. © 2008 American Institute of Physics.

I. INTRODUCTION

Density peaking in tokamak plasmas has been shown to decrease with increasing collisionality in ASDEX Upgrade and JET (Joint European Torus) H-modes. These experimental results suggest that the particle transport, which is usually dominated by ITG- and TEM-driven turbulence, depends on the collisionality, although it has been suggested that the source from ionization may also play an important role.

In the collisionless limit, numerical simulations of ITG-mode driven turbulence give an inward particle flux in fluid, gyrofluid, and gyrokinetic descriptions. The inward flow is caused mainly by magnetic curvature and thermodiffusion. However, nonlinear gyrokinetic calculations show that even a small value of the collisionality affects strongly the magnitude and sign of the anomalous particle flows. The inward particle flow obtained in the collisionless limit is rapidly converted to outward flow as electron-ion collisions are included. Linear gyrokinetic calculations with GS2 and a quasilinear model for the particle fluxes have confirmed the strong collisionality dependence of the quasilinear particle flux for small collisionalities and show a good agreement with nonlinear gyrokinetic results from Ref. 8. Both gyrokinetic models find that the total particle flux becomes directed outward for much smaller values of collisionality than the lowest collisionality presently achieved in tokamaks. This means that according to these simulations, for present tokamak experiments the particle flow should be outward.

The present paper addresses the collisionality dependence of the quasilinear flux due to ITG and TEM modes. The aim is to derive analytical expressions for the quasilinear flux to show explicitly the dependence on collisionality, density, and temperature gradients, so that the sign and magnitude of the flux can easily be estimated. We focus on the collisionality dependence of the direction and the magnitude of the quasilinear flux, and give approximate analytical expressions for weakly collisional plasmas with large aspect ratio and circular cross section.

The collisionality dependence has previously been studied in Refs. 10 and 11 by approximating the collision operator with an energy-dependent Krook operator. The main difference between this paper and the references above is the form of the collision operator. Here we use a pitch-angle scattering collision operator, but we include the results for the Krook operator for comparison and completeness. As we will show here, the form of the collision operator determines the scaling with collisionality and therefore affects the collisionality threshold at which the particle flow reverses. The eigenfrequency and growth rate of the modes are only weakly dependent on the collisionality, and in this paper we do not analyze the dispersion relation and the stability boundaries, but instead focus on the quasilinear particle flux driven by the mode. The collisionality dependence of the quasilinear flux due to the TEM instability has been studied in Ref. 12 using a pitch-angle scattering collision operator, and here we generalize the expression presented there by including the magnetic drift.

The structure of the paper is the following: In Sec. II the general gyrokinetic formalism is presented. In Sec. III approximate solutions of the gyrokinetic equation are given in the limit of high mode numbers and the perturbed electron density is calculated. In Sec. IV the quasilinear flux is calculated and the effect of collisions is discussed. The possibility of flux reversal, comparison with previous work and the validity of our approximations are discussed in Sec. V. Finally, the results are summarized in Sec. VI.
II. GYROKINETIC EQUATION

We consider an axisymmetric, large aspect ratio torus with circular magnetic surfaces. The nonadiabatic part of the perturbed distribution function is given by the linearized gyrokinetic equation \(^{(0)}\)

\[
\frac{v_i}{qR} \frac{\partial g_0}{\partial \theta} - i(\omega - \omega_{si})g_0 - C_0(g_0) = 0,
\]

where \(\theta\) is the extended poloidal angle, \(\phi\) is the perturbed electrostatic potential, \(f_{e0} = n_e (m_e/2) \pi T_e)^{3/2} \exp(-w/T_e)\) is the equilibrium Maxwellian distribution function, \(w = m_e v^2/2, \) \(n_e, T_e,\) and \(\epsilon_e\) are the density, temperature, and charge of species \(a,\) respectively. \(\omega_{si} = -k \tau_{aB}/eB_L\) is the diamagnetic frequency, \(\omega_{ce} = n_e [1 + (w/T_e - 3/2)] / \eta,\) \(\eta_a = L_a / L_T,\) \(L_{ao} = \pm |(\ln n_a)/\partial r|^{-1}, L_{ao} = \pm |(\ln T_a)/\partial r|^{-1},\) are the density and temperature scale lengths, respectively, \(k_0\) is the poloidal wave-number, \(\omega_{de} = -k_0 (v_e^2 + v_i^2) / \cos \theta + s \sin \theta / \omega_{si}\) is the magnetic drift frequency, \(\omega_{si} = eB / m_e\) is the cyclotron frequency, \(q\) is the safety factor, \(s = r/q (dq/dr)\) is the magnetic shear, \(r\) is the minor radius, \(R\) is the major radius, \(J_0\) is the Bessel function of order zero, and \(\zeta_a = k_0 v_i / \omega_{si} = k_0 v_i / \omega_{si}\).

III. PERTURBED ELECTRON DENSITY RESPONSE

Turning to the electron kinetic equation, we retain collisions and use a pitch-angle scattering operator

\[
C_0 = \frac{v_i}{B} \frac{2 \xi}{\partial \theta} \frac{\partial}{\partial \theta} \xi = \frac{v_i}{B} \xi,
\]

where \(v_i = v_i \xi / \lambda,\) \(\xi = v_i / v,\) and \(\lambda = \mu / w\) with \(\mu = m_e v_i^2 / 2B.\) If the electron distribution is expanded as \(g_0 = g_{e0} + g_{e1} + \cdots\) in the smallness of the normalized collisionality \(v_{ei} = v_i / \omega_{si} \ll 1\) and \(\omega / \omega_{si} \ll 1,\) where \(\omega_{si}\) is the bounce frequency, then in lowest order we have \(\partial g_0 / \partial \theta = 0.\) In next order, we arrive at

\[
i(\omega - \omega_{si})g_0 + (C_0 g_{e0}) = \frac{i e(\phi)}{T_e} (\omega_{ce} - \omega) f_{e0},
\]

where \(\langle \cdot \rangle\) is the orbit average. The circulating electrons are assumed to be adiabatic, while in the trapped region \(g_{e0}\) is given by

\[
(\omega - \omega_{si})g_{e0} = \frac{2 i v_e \sqrt{2} e}{\tau_{eB}} \frac{\partial}{\partial \lambda} \left( \int \frac{\xi d\theta}{\partial \theta} \frac{\partial g_{e0}}{\partial \lambda} \right)
\]

\[
= \frac{e(\phi)}{T_e} (\omega - \omega_{te}) f_{e0},
\]

where the orbit-averaged precession frequency for trapped electrons is

\[
\langle \omega_{de} \rangle = \omega_{de} \left[ E(\kappa) / K(\kappa) - 1 + \frac{2 q T_e}{q} E(\kappa) / K(\kappa) + \kappa - 1 \right],
\]

where \(\omega_{de} = -k_0 v_i / \omega_{si} R,\) and \(E, K,\) and \(\kappa\) are the complete elliptic functions with the argument \(\kappa = (1 - \lambda B_0 (1 - e))/2eB_0,\) where \(B_0\) is the flux-surface averaged magnetic field and \(\epsilon = r/R.\) Performing the orbit average on the scattering operator, Eq. (4) becomes

\[
(\omega - \omega_{si})g_{e0} = \frac{i v_e}{e^2} \frac{\partial}{\partial \lambda} \left[ \frac{J(\kappa)}{T_e} \frac{\partial g_{e0}}{\partial \lambda} \right] = \frac{e(\phi)}{T_e} (\omega - \omega_{te}) f_{e0},
\]

where \(J = E(\kappa) + (\kappa - 1) K(\kappa)\) and \(\gamma_{te} = K(\kappa).\) We introduce a parameter \(\gamma = v_i / \omega_{si},\) where \(\omega_{si} = \omega / \gamma,\) \(\gamma = \sigma + i \gamma,\) \(\sigma = \text{sign} \left( \text{Re} \left( \omega_{si} \right) \right)\) denotes the sign of the real part of the eigen-frequency, and \(\gamma = \gamma / \omega_t\) is the normalized growth rate. The equation for \(g_{e0}\) is

\[
\dot{g}_{e0} + (\ln J) g_{e0} + i \frac{K}{J} \left( y - \frac{\omega_{si}}{\omega_{te}} \right) g_{e0} = i \frac{SK}{\omega_{si}} f_{e0},
\]

where \(S = -(e(\phi)/T_e)(\omega - \omega_{te}) f_{e0}.\) The perturbed electrostatic potential is approximated by \(\phi(\theta) = \phi_0 (1 + \cos \theta)/2[H(\theta + \pi) - H(\theta - \pi)],\) where \(H\) is the Heaviside function and \(\phi = \phi_0 E(\kappa) / K(\kappa).\) Assuming weakly collisional plasmas such that \(\gamma \ll 1,\) the WKB solution to the homogeneous equation \(\dot{g}_{e0} + (\ln J) g_{e0} = \Omega^2 g_{e0},\) where \(\Omega^2 = -(\omega_{de} / \omega_t) K(\kappa)/J(\kappa),\) is

\[
g_{e0}(\kappa) = \frac{1}{\sqrt{\Omega}} \left[ c_1 \sinh \left( \sqrt{\pi} \int_{\xi}^{\kappa} \Omega(z) dz \right) + c_2 \cosh \left( \sqrt{\pi} \int_{\xi}^{\kappa} \Omega(z) dz \right) \right].
\]

The solution of the inhomogeneous equation can then be obtained with the method of variation of parameters, using the boundary conditions at \(\kappa = 0\) and \(\kappa = 1\) to determine the integration constants \(c_1\) and \(c_2.\)

To make further progress analytically, we need to approximate the elliptic functions with their asymptotic limits for small argument, as was done in Ref. 12, so that \(K(\kappa)/J(\kappa) = 2/\kappa.\) The homogeneous solution becomes

\[
g_{e0}(\kappa) = \frac{1}{(\kappa u)^{1/4}} \left[ c_1 \sinh (2 \sqrt{\kappa u} \rho) + c_2 \cosh (2 \sqrt{\kappa u} \rho) \right],
\]

where \(u = -i(2y - \omega_{de}),\) and \(\omega_{de} = \omega_{de} / \omega_t\) is the normalized magnetic drift frequency. The inhomogeneous part of the distribution is given by

\[
g_{e0}(\kappa) = \frac{2 S}{u \omega_t} \left[ 1 - \frac{\sqrt{\pi}}{4} e^{z^2 - z^2 e} \text{Erf}(z) \right],
\]

where \(z = \sqrt{\kappa u} \rho \gamma_{te}^{1/4}, \) \(S = S(\kappa) / \kappa,\) \(\text{Erf}(z), \text{Erf}(z)\) is the error function, and \(\text{Erf}(z) / \text{Erf}(z) = i / \gamma_{te}^2\) is the imaginary error function. In the limit of \(z \rightarrow \infty\) (consistent with the assumption \(\gamma_{te} \ll 1),\) the error functions can be expanded, and the inhomogeneous solution simplifies to
Since $g_{\text{el}}(\kappa=0)$ is regular, we choose $c_2=0$, and the boundary condition $g_{\text{el}}(\kappa=1)=0$ (Ref. 12) gives $c_1=-g_{\text{memb}}(1)\nu^3/\sinh(2\sqrt{u}/\hat{v})$, so that the solution for the perturbed trapped-electron distribution is

$$
g_{\text{el}} = -\frac{i\hat{S}(4-\sqrt{\pi}e^2/z)}{2u\omega_0} + \frac{i\hat{S}(4-\sqrt{\pi}e^2z/\kappa)}{2u\kappa^{\frac{3}{2}}\omega_0} \sinh(\kappa z/\kappa),$$

except in a narrow boundary layer close to $\kappa=0$, that has a negligible contribution to the velocity space integrals. The nonadiabatic part of the perturbed trapped-electron density is proportional to

$$
\left\langle \int g_{\text{el}}d^3v \right\rangle = 4\sqrt{2}\epsilon^2 \int_0^1\int_0^{\pi/2} K(\kappa)g_{\text{el}}d\kappa = 4\sqrt{2}\epsilon^2 \int_0^1\int_0^{\pi/2} \kappa(\nu-u\hat{v})d\kappa,
$$

Using Eq. (12), the integral $I_1$ becomes

$$
I_1 = \int_0^{\pi/2} K(\kappa)g_{\text{el}}d\kappa = -\frac{\pi\hat{S}}{\nu^3\omega_0}(\nu-u\hat{v}),
$$

where we retained terms only to the lowest order in $\nu^{1/2}$ and we approximated $K(\kappa) = \pi/2$.

The above analysis is not valid in the boundary layer at $\kappa=1$. The effect of the boundary layer reduces the collisional term, with a factor $\pi/2\sqrt{\nu}$ (see Appendix A for details). The reduction is less than 20% in the experimentally relevant collisionality regime, and in the following analysis this will be neglected.

Expanding in the limit of small $\hat{\omega}_{\text{el}}$ and keeping terms to the first order, we have

$$
2I_1 = \frac{\pi}{8} \left[ 2\hat{S}(y^2 + 3\hat{v}_e^2) + 4y(\hat{\omega}_{\text{el}} - \hat{v}_e) \right].
$$

Introducing $\hat{\omega}_{\text{el}}=\hat{\omega}_{\text{el}}/\kappa^2$, $\hat{\omega}_{\text{el}}=\omega_{\text{el}}/\omega_0$, and $\hat{v}_e=\hat{v}_e^3$, we obtain

$$
\left\langle \int g_{\text{el}}d^3v \right\rangle = \frac{\pi}{8} S_0 \left[ \frac{1-\hat{\omega}_{\text{el}}}{\nu^3} + \frac{3\hat{\omega}_{\text{el}}}{4y} \left( 1 - (1 + \eta_e) \frac{\hat{\omega}_{\text{el}}}{\nu^3} \right) \right],
$$

$$
\frac{-i\pi}{8} \Gamma(3/4)(3/\nu^3)\hat{v}_e + \frac{3\hat{\omega}_{\text{el}}}{4y} \left( 1 + (3 - 4 \eta_e - 4) \frac{\hat{\omega}_{\text{el}}}{4y} + \frac{9\hat{\omega}_{\text{el}}}{64y} \right) \times \left( 4 - (4 + \eta_e) \frac{\hat{\omega}_{\text{el}}}{\nu^3} \right),
$$

where $S_0=(e\phi/T_e)n_A^2(2e/\pi)^{3/2}$. Neglecting the nonadiabatic circulating electron response, the perturbed electron density is

$$
\hat{n}_e = \frac{e\phi}{2T_e} \frac{\hat{v}_e}{\nu^3} \left( 1 - 2\epsilon \frac{1 - \hat{\omega}_{\text{el}}}{\nu^3} + \frac{3\hat{\omega}_{\text{el}}}{4y} \right) \times \left( 1 - (1 + \eta_e) \frac{\hat{\omega}_{\text{el}}}{\nu^3} \right) \times \left( 1 + (3 - 4 \eta_e - 4) \frac{\hat{\omega}_{\text{el}}}{4y} + \frac{9\hat{\omega}_{\text{el}}}{64y} \right) \times \left( 4 - (4 + \eta_e) \frac{\hat{\omega}_{\text{el}}}{\nu^3} \right).
$$

IV. QUASILINEAR PARTICLE FLUX

The quasilinear particle flux is given by

$$
\Gamma_e = \text{Re}(\hat{n}_e v_e^f) = \frac{e\phi_0}{2eB} \frac{v_e}{T_e} \left\langle \int \frac{\hat{n}_e}{\nu^3} \right\rangle,
$$

where the radial $E\times B$ velocity is $v_e=-ik\phi_0/2B$. Taking the imaginary part of the perturbed electron density from Eq. (17), we obtain

$$
\text{Im} \left\langle \frac{\hat{n}_e}{\nu^3} \right\rangle = \frac{-i\pi}{8} S_0 \left[ \frac{1-\hat{\omega}_{\text{el}}}{\nu^3} + \frac{3\hat{\omega}_{\text{el}}}{4y} \left( 1 - (1 + \eta_e) \frac{\hat{\omega}_{\text{el}}}{\nu^3} \right) \right],
$$

$$
\frac{-i\pi}{8} \Gamma(3/4)(3/\nu^3)\hat{v}_e + \frac{3\hat{\omega}_{\text{el}}}{4y} \left( 1 + (3 - 4 \eta_e - 4) \frac{\hat{\omega}_{\text{el}}}{4y} + \frac{9\hat{\omega}_{\text{el}}}{64y} \right) \times \left( 4 - (4 + \eta_e) \frac{\hat{\omega}_{\text{el}}}{\nu^3} \right).$$

The imaginary part of the perturbed density is sensitive to the sign of the real part of the eigenfrequency $\sigma$, and the magnitude of the normalized growth rate $\gamma$. In the following analysis the quasilinear flux will be calculated for negative (ITG) and positive (TEM) signs.

A. ITG

ITG modes propagate in the ion diamagnetic direction, so the real part of the eigenfrequency is negative. Figure 1 shows the quasilinear electron flux from Eqs. (18) and (19) normalized to $p_{ke}^{1/2}(e\phi_0/T_e)^{1/2}$ as function of normalized collisionality for various values of $\hat{\omega}_{\text{el}}$ and $\eta_e$ for a case where the plasma is far from marginal stability: $\gamma=0.7$.

In the absence of collisions, the flux is inwards if the curvature and thermodiffusive fluxes (the terms proportional to $\hat{\omega}_{\text{el}}$ and $\eta_e$ in the first row of Eq. (19)) dominate over diffusion. If collisions are included, the particle flux may be reversed, if the part of the flux that is dependent on the collisionality is positive. This reversal happens, for instance, for $\hat{\omega}_{\text{el}}=0.2$ and $\eta_e=4.5$ [see Fig. 1(d)].
However, if the ITG-instability growth rate is weak ($\gamma \ll 1$) and $\eta_\parallel$ is large, the situation is completely different. Figure 2 shows the normalized quasilinear electron flux for the same parameters as in Fig. 1, but for $\gamma=0.1$, representing a case close to marginal stability. The term proportional to the $\sqrt{\nu_i}$ will change sign and now this will also lead to an inward flux. If the magnetic drift is high enough to give an inward flux for zero collisionality, then collisions will enhance this and the flux will therefore never be reversed. If the magnetic drift is very small, the flux is outwards for $\nu_i = 0$. Then collisions may reverse the sign of the flux, but now from outwards to inwards.

It is instructive to expand Eq. (19) for small $\gamma$, and show explicitly the sign of the different terms in the expression for the flux. If $y=\omega/\omega_0=-1+i\gamma$, then to lowest order in $\gamma$, we have

\[ \hat{f}_e = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx \frac{\omega_0}{\omega} \left( 1 + \frac{\gamma}{\sqrt{\nu_i}} \right) \]

FIG. 1. Normalized quasilinear electron flux driven by ITG as function of normalized collisionality for $\gamma=0.7$ and $\omega_0=1$. In the upper figures (a and b): from above $\omega_0=0$ (solid), 0.1 (long-dashed), 0.2 (short-dashed), and 0.4 (dotted). (a) $\eta_\parallel=3$ and (b) $\eta_\parallel=6.5$. In the lower figures: $\eta_\parallel$ is 3 (solid), 4.5 (long-dashed), 6.5 (short-dashed), and 8.5 (dotted). (c) $\omega_0=0$ and (d) $\omega_0=0.2$.

FIG. 2. Same as Fig. 1, but for $\gamma=0.1$. 

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If the plasma is close to marginal instability ($\gamma = 0$), collisions (represented by the term proportional to $\sqrt{\nu_i}$) lead to an inward flux if $\eta_i > \eta_{crit}$:

$$\eta_{crit} = \frac{4[16(1 + \omega_x) - 9\dot{\omega}_{DI}(1 + \omega_x)]}{3(16 + 3\dot{\omega}_{DI})\omega_x}.$$  

For typical experimental parameters, $\eta_i$ is expected to be larger than $\eta_{crit}$ and therefore the total flux is expected to be inwards. However, if the plasma is further away from marginal instability, so that $\gamma > 2/3$, the terms to $1-3\gamma/2$ and $1-5\gamma/2$ change sign, and then collisions will lead to an outward flux, as Fig. 1 shows. Note that the figures show the quasilinear flux calculated from the unexpanded solution (Eqs. (14) and (18)) and they are valid even for $\gamma = 1$.

### B. TEM

The real part of the eigenfrequency is positive, and this means that $\gamma = \omega/\omega_0 = 1 + i\hat{\gamma}$ and the electron flux to lowest order is

$${\hat{f}}_e = k_0|q_e| \frac{e\phi_0}{T_e} \left\{ \frac{2}{\sqrt{\pi}} \int_0^\infty \left[ 1 - \frac{3\dot{\omega}_{DI}}{2}(1 + \eta_i) \right] \hat{\gamma}\omega_x \right. \right.$$

$$- \sqrt{\frac{\epsilon}{2}} \frac{3\dot{\omega}_{DI}\hat{\gamma}}{4} - \Gamma(3/4)\frac{\sqrt{\epsilon \nu_i}}{2\pi} \left\{ -1 + \hat{\gamma} + \hat{\omega}_{x} \right\}$$

$$\times \left( 1 - \frac{3\hat{\gamma}}{2} + \frac{9\dot{\omega}_{DI}}{16} \left[ 1 - \hat{\gamma} + \frac{1 + \eta_i}{4} \right] \right)$$

$$\times \left( 1 - \frac{5\hat{\gamma}}{2} \hat{\omega}_{x} \right) \left\} \right\}.$$  

There are two main differences compared with the ITG-driven flux. First, the part of the flux that is driven by the curvature has opposite sign compared with ITG, and therefore contributes to the outward flux instead of driving an inward pinch. Second, the part of the flux that arises due to collisions is different and may have opposite sign compared with the ITG case, depending on the parameters. Figure 3 shows the normalized quasilinear flux for different parameters if the plasma is far from marginal stability ($\gamma = 0.7$) and Fig. 4 shows the same for $\gamma = 0.1$. Also here, the magnitude of $\gamma$ changes the sign of the flux from outward to inward, and collisions contribute to the inward flux.

### C. Collisions modeled by a Krook operator

Starting from the gyrokinetic equation for the electrons but modeling the collision operator with an energy-dependent Krook operator, we have

\[ \frac{\partial f}{\partial t} + \hat{v} \cdot \nabla f = -\frac{1}{\epsilon} \frac{\partial}{\partial \epsilon} \left( \hat{v} \cdot \nabla \epsilon f \right) + \frac{1}{\epsilon} \int_{\mathbf{k}'} D_{\epsilon} f_{\epsilon'} f_{\epsilon'}(\hat{v} - \hat{v}') d\hat{v}' \]

where $D_{\epsilon}$ is the Krook collision operator.

In the ITG case, the Krook operator is given by

\[ D_{\epsilon} = \frac{1}{\epsilon} \int_{\mathbf{k}'} \frac{\partial f}{\partial \epsilon} f_{\epsilon'} f_{\epsilon'}(\hat{v} - \hat{v}') d\hat{v}' \]

and in the TEM case, the Krook operator is given by

\[ D_{\epsilon} = \frac{1}{\epsilon} \int_{\mathbf{k}'} \frac{\partial f}{\partial \epsilon} f_{\epsilon'} f_{\epsilon'}(\hat{v} - \hat{v}') d\hat{v}' \]

These equations describe the evolution of the electron distribution function under collisional effects, where the Krook operator accounts for the energy transfer due to collisions.
De_{\text{eff}} = -e_{0} T_{e} - e_{T} f_{e0}^{*} + \frac{i}{e_{0}} \text{eff}

so that

\begin{equation}
\Phi_{e} = -\frac{e_{0}}{T_{e}} (\omega - \omega_{se}) f_{e0}
\end{equation}

where \( r_{\text{eff}} \equiv v_{T}/\epsilon_{3}^{3} \).

The velocity-space integral of the perturbed electron distribution can be used to determine the imaginary part of the perturbed electron density, and that gives the quasilinear flux from Eq. (18). If the plasma is far from marginal stability, the results for the pitch-angle scattering and Krook operator are qualitatively same, as shown in the upper figures of Fig. 5. However, as the lower figures in Fig. 5 show, as we approach marginal stability, the form of the collision operator matters more and more, and both the sign and the magnitude of the flux may be very different.

V. DISCUSSION

We have shown that in plasmas that are dominated by ITG turbulence and are far from marginal stability the sign of the electron flux will be changed from inward to outward if the collisionality is increased. Figure 6 shows the threshold in collisionality for which the flux reverses, i.e., \( \hat{\nu}_{e} \), as a function of \( \eta_{e} \) for different values of the normalized magnetic drift frequency. The red curves correspond to the pitch-angle scattering model-operator and the black curves correspond to the Krook model. It is interesting to see that the pitch-angle scattering operator gives lower threshold for flux reversal. If we compare the collisionality for zero flux in Fig. 4 in Ref. 8, we find that our result is of the same order of magnitude. Figure 4 in Ref. 8 is computed for \( \eta_{e}=3 \), \( R=3 \) m, \( r=0.5 \) m, \( a=1 \) m, \( s=1 \), \( q=2 \). For these parameters the trapped-electron flow changes sign for \( \hat{\nu}_{e}=0.006c_{s}/a \), where \( c_{s} = \sqrt{T_{e}/m_{i}} \) is the ion sound speed. This collisionality corresponds to \( \hat{\nu}_{e}=0.1 \). This is in agreement with our threshold, shown in Fig. 6 for \( \eta_{e}=3 \) and \( \hat{\omega}_{Dn}=0.7 \). Note that Fig. 4 in Ref. 8 is the result of a nonlinear gyrokinetic simulation so \( \hat{\omega}_{Dn} \) is not constant and therefore exact comparison is not possible.

The analytical calculation presented in this paper is an attempt to shed light on the numerical calculations mentioned above. For this purpose it is necessary to make a number of simplifications, some of which may be justified within a rigorous ordering scheme. However, it should be noted that some approximations are more qualitative, in particular those having to do with the mode structure, which we do not solve for. The approximation we use for the perturbed electrostatic potential breaks down for low shear or near marginal instability. It appears that the qualitative features of the transport are captured by our calculations, but for quantitatively accurate results one of course has to resort to numerical simulations.

As we have seen, the effect of magnetic drift is important to understand the sign change of the quasilinear flux due to the ITG modes. The magnetic drift gives an inward flux for zero collisionality, but this is reversed when \( \hat{\nu}_{c} > \hat{\nu}_{c} \).

VI. CONCLUSIONS

The collisionality dependence of the quasilinear particle flux due to microinstabilities has been determined for large aspect ratio, circular cross section plasmas. It has been shown that if the plasma is far from marginal stability, the
inward transport due to ITG modes is reversed as electron collisions are introduced, in agreement with nonlinear gyrokinetic simulations. However, if the plasma is close to marginal stability, collisions will lead to an additional inward flux, and therefore the total flux is expected to be inwards. The transport is therefore affected significantly by the parameter $\eta_e$, both directly via the terms proportional to $\eta_e$ in the expression for the flux, but also indirectly via the ITG growth rate, which is important to determine the sign of the flux.

If the electron collisions are modeled with a pitch-angle scattering collision operator, the particle flux is proportional to the square-root of the collisionality. The choice of the model collision operator affects the collisionality threshold for the reversal of the particle flux; i.e., $\hat{\eta}_c$. This is especially important when the plasma is close to marginal stability. The collisionality threshold $\hat{\eta}_c$ depends on the magnitude of the normalized magnetic drift $\hat{\delta}_{Dm}$ and the ratio of density and temperature scale lengths; i.e., $\eta_e$. For higher $\eta_e$ and higher $\hat{\delta}_{Dm}$, higher collisionality is needed to reverse the particle flux.

The magnitude and the sign of the TEM-driven quasilinear flux has also been determined. The TEM-driven flux is expected to be outwards if the plasma is far from marginal stability and inwards otherwise, for typical experimental parameters, and the presence of collisions contributes to the inward flow.

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APPENDIX A: BOUNDARY LAYER ANALYSIS FOR $\kappa=1$

In the outer region, far away from the trapped/passing boundary we can neglect collisions, and the solution of Eq. (7) is
Changing variables in Eq. \( H_9260 \) distribution has been calculated to be \( \mu_72308-8 \) Fülöp, Pusztai, and Helander Phys. Plasmas 15 \( \frac{i \hat{K}}{\hat{y}} \) gives

\[
\frac{\partial^2 g_{\text{inner}}}{\partial \hat{y}^2} + i \hat{K} \left( y - \frac{\langle \omega_{Dy} \rangle}{\omega_0} \right) g_{\text{inner}} = i \frac{\hat{S}}{\omega_0}, \tag{A2}
\]

and the solution is

\[
g_{\text{inner}}(y) = \frac{\hat{S}}{(\omega - \langle \omega_{Dy} \rangle) \hat{K}} \left( 1 - \exp\left[ - (1 - \kappa) \sqrt{\hat{u} \hat{K} / \hat{v}} \right] \right), \tag{A3}
\]

where \( \hat{u} = -i(y - \langle \omega_{Dy} \rangle / \omega_0) \). \( \hat{c}_1 \) is determined by the boundary condition \( g_{\text{inner}}(\kappa=1) = 0 \) and \( \hat{c}_2 = 0 \) to match the inner and outer solutions. The global solution is then

\[
g_{\text{outer}} = i \frac{\hat{S}}{(\omega - \langle \omega_{Dy} \rangle) \hat{K}} \left[ 1 - \exp\left[ - (1 - \kappa) \sqrt{\hat{u} \hat{K} / \hat{v}} \right] \right]. \tag{A4}
\]

Using the global solution from Eq. (A4), the collisional term becomes

\[
\frac{1}{\kappa} K(\kappa) \frac{\hat{S}}{(\omega - \langle \omega_{Dy} \rangle) \hat{K}} \exp\left[ - (1 - \kappa) \sqrt{\hat{u} \hat{K} / \hat{v}} \right] d\kappa
\]

\[
= - \frac{\hat{S}}{(\omega - \langle \omega_{Dy} \rangle / 2) \sqrt{\hat{K}}} \sqrt{\hat{v}} \tag{A5}
\]

since the dominant part of the integral comes from \( \kappa = 1 \). Comparing with the corresponding term in Eq. (14), we find that the effect of the boundary layer reduces the collisional term, with a factor \( (\pi / 2) \ln \hat{v}^{1/2} \).

**APPENDIX B: COMPARISON WITH THE SOLUTION FOR \( \omega_0 T_\gamma = 0 \) IN REF. 12**

In Ref. 12, the effect of the magnetic drift has been neglected (that is, \( \omega_0 T_\gamma = 0 \)), and the perturbed trapped electron distribution has been calculated to be

\[
g_{\text{coll}} = - \frac{e \phi_0}{T_e} \left( 1 - \omega_{Dy} / \omega_0 \right) f_0 \left[ 1 - \frac{2 J_1(\alpha)}{\alpha J_0(\alpha)} \right]. \tag{B1}
\]

where \( \alpha = (1 + i) \sqrt{4 \omega e \nu_e}(v) = a_i(1 + i) \). There is excellent agreement between our results in the limit of \( \omega_0 T_\gamma = 0 \) (both the expanded solution and the full WKB solution) and the one published in Ref. 12. The perturbed electron density is

\[
\frac{\dot{n}_e}{n_e} = e \phi_0 \frac{T_e}{\omega_0} \left[ 1 - \frac{1}{2} \frac{\omega_{Dy}}{\omega_0} \left( 1 - \eta(x^2 - 3/2) \right) \right]. \tag{B2}
\]

If the real part of the frequency is negative \( (\omega = -\omega_0 + i\gamma) \), where \( \omega_0 > 0 \), then for \( \alpha_i > 1 \) and \( \gamma \ll 1 \), we have

\[
\left| \frac{\dot{\gamma}/n_e}{e \phi_0 / T_e} \right| = \frac{2 \sqrt{2e}}{\pi} \omega_{Dy} \gamma + \left[ \frac{1}{3} + \omega_{Dy}(1 - 3 \eta(x^2 - 4)) \right]. \tag{B3}
\]

This is in agreement with our results in the limit of \( \omega_0 T_\gamma = 0 \).

Paper B

Collisonal model of quasilinear transport driven by toroidal electrostatic ion temperature gradient modes

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The stability of ion temperature gradient (ITG) modes and the quasilinear fluxes driven by them are analyzed in weakly collisional tokamak plasmas using a semianalytical model based on an approximate solution of the gyrokinetic equation, where collisions are modeled by a Lorentz operator. Although the frequencies and growth rates of ITG modes far from threshold are only very weakly sensitive to the collisionality, the \( a/L_T \) threshold for stability is affected significantly by electron-ion collisions. The decrease in collisionality destabilizes the ITG mode driving an inward particle flux, which leads to the steepening of the density profile. Closed analytical expressions for the electron and ion density and temperature responses have been derived without expansion in the smallness of the magnetic drift frequencies. The results have been compared with gyrokinetic simulations with GYRO and illustrated by showing the scalings of the eigenvalues and quasilinear fluxes with collisionality, temperature scale length, and magnetic shear. © 2009 American Institute of Physics. [DOI: 10.1063/1.3168611]

I. INTRODUCTION

Turbulent transport in tokamak plasmas is considered to be mainly caused by drift waves destabilized by trapped electrons and ion temperature gradients.1–4 These microinstabilities and their effect on the transport can be studied by complex nonlinear gyrokinetic codes, for example GYRO.5 To ease the interpretation of the results of these codes and experimental results it is useful to construct simpler models that can, after careful benchmarking with codes, give various parametric scalings. In particular, the collisionality dependence of the microinstabilities is interesting from both experimental and theoretical points of view. On the experimental side, the evolution of the density profile has been shown to depend on the collisionality.6–10 On the theoretical side, it has been shown that the transport fluxes are dependent on the choice of the collision operator.11

Numerical simulations of ion temperature gradient (ITG) and trapped electron (TE) modes have shown that collisions may influence the sign and the magnitude of the quasilinear fluxes driven by these instabilities.11 Without collisions, the quasilinear particle flux driven by ITG modes is usually inward due to curvature and thermodiffusion. Gyrokinetic calculations show that collisions drive an outward flux and the particle flux is expected to change sign for very small collisionalities, much smaller than the collisionality achievable in current tokamak experiments. The choice of the model collision operator affects the collisionality threshold for the reversal of the particle flux.11 This means that collisionless models or models using the Krook model operator are not adequate to calculate the quasilinear transport fluxes for typical experimental parameters.

In this work we develop a collisional model for electrostatic turbulence (COMET) that can be used to analyze the stability of the ITG modes and to derive analytical expressions for the quasilinear fluxes. Much of the theoretical analysis of the effect of collisions on ITG modes has been based on an energy-dependent Krook operator.13–15 Here, we model the collisions by a Lorentz operator, which automatically incorporates the increasing importance of pitch-angle scattering near the trapped-passing boundary.16 We focus on weakly collisional plasmas with large aspect ratio and circular cross section and retain the effect of the magnetic drift nonperturbatively.

The collisionality dependence of the particle flux driven by microinstabilities has been studied in Ref. 11 under the assumption that the mode frequency and growth rate are independent of the collisionality. In this work, the model presented in Ref. 11 is extended to include the effect of collisions on the eigenfrequency, growth rate, and stability boundaries of the modes. We show that far from marginal stability the collisionality dependence of the ITG eigenfrequency and growth rate is weak and therefore will have a negligible effect on the particle fluxes. However, we found that the \( a/L_T \) stability threshold is sensitive to the electron-ion collisions. We found an exact ITG stability boundary in the adiabatic limit which incorporates the shear and the finite Larmor radius (FLR) parameter dependences.

To determine the perturbed electrostatic potential self-consistently would involve the solution of an integrodifferential equation that is analytically intractable. Therefore, as in Ref. 11, we use a model electrostatic potential, valid in the moderate shear region and motivated by a variational method. The model potential used here is improved compared to the one used in Ref. 11, by including a shear dependent imaginary part of the potential. Assuming this balloon-
ing model potential, closed analytical expressions for the electron and ion density and temperature perturbations are derived, without expansion in the smallness of the magnetic drift. These are used to compute the quasilinear particle and energy fluxes. The results of COMET are benchmarked with numerical gyrokinetic simulations with GYRO and are useful to show the scalings with collisionality, magnetic drift frequency, diamagnetic frequency, and ratio of the density and temperature scale lengths.

The remainder of the paper is organized as follows. In Sec. II, the perturbed electron and ion density and temperature responses are calculated. In Sec. III the dispersion relation is presented and the dependence of the stability boundaries on collisionality is studied. In Sec. IV the quasilinear transport fluxes are calculated and scalings of the growth rates, eigenfrequencies, and fluxes with temperature scale length, collisionality, and magnetic shear is discussed and compared with gyrokinetic simulations with GYRO. Finally, the results are summarized in Sec. V.

II. PERTURBED ELECTRON AND ION RESPONSES

The perturbed electron and ion responses are obtained from the linearized gyrokinetic (GK) equation,

$$\frac{v_i}{qR} \frac{\partial g_a}{\partial \theta} - i(\omega - \omega_{D0})g_a - C_a(g_a) = -i \frac{e_{f_i}}{T_a} \frac{\partial}{\partial \omega} \left( \frac{\omega - \omega_{a}}{\omega_{a}} \right) \phi_J(z_a),$$

where $g_a$ is the nonadiabatic part of the perturbed distribution function, $\theta$ is the extended poloidal angle, $\phi$ is the perturbed electrostatic potential, $f_{i0} = n_i/(\sqrt{4\pi} \omega_{pi}) \exp(-x_i^2)$ is the equilibrium Maxwellian distribution function, $x_i = v_i/v_{Te}$ is the velocity normalized to the thermal speed $v_{Te} = (2T_e/m_e)^{1/2}$, $n_i$, $T_i$, $m_i$, and $e_i$ are the density, temperature, mass, and charge of species $i$, $\omega_{a} = -k_B T_e/\varepsilon B L_{mn}$ is the diamagnetic frequency, $\omega_{f_i} = \omega_{a} (1 + [x_i^2 - (3/2)] \eta_i)$, $\eta_i = L_{mn}/L_{Te}$, $L_{mn} = [\partial \ln n_i/\partial r]^{-1}$, $L_{Te} = [\partial \ln T_e/\partial r]^{-1}$, are the density and temperature scale lengths, $k_B$ is the poloidal wave number, $\omega_{D0} = -k_B (v_i^2/2 + e_i^2) \cos \theta + s \sin \theta/\omega_{D0} R$ is the magnetic drift frequency, $\omega_{eo} = e B/m_e$ is the cyclotron frequency, $B$ is the equilibrium magnetic field, $q$ is the safety factor, $s = (r/q) (d \alpha/\partial r)$ is the magnetic shear, $r$ and $R$ are the minor and major radii, $J_0$ is the Bessel function of order zero, and $z_a = k_B v_{Te} / \omega_{eo}$. We consider an axisymmetric, large aspect ratio torus with circular magnetic surfaces. We adopt the usual ordering for the relation of the electron/ion bounce frequencies and the eigen-frequency of the mode $\omega_{eo} << \omega \ll \omega_{he}$ and we consider weakly collisional plasmas so that $v_n = v_n/\nu_{he} < 1$, where $v_n$ is the electron-ion collision frequency and $\nu_e = r/R$ is the inverse aspect ratio. The ion self-collisions and ion-electron collisions are neglected $[C_i(g_i) = 0]$, while the electron-ion collisions are modeled by a pitch-angle scattering operator.

A. Electron response

The circulating electrons are assumed to be adiabatic. The nonadiabatic electron distribution can be expanded $g_e = g_{e0} + g_{e1} + \cdots$ in the smallness of $\omega/\omega_{he}$ and the normal-
ized collisionality $\nu_0$, which gives $\partial g_{\text{el}}/\partial \theta=0$ in lowest order. The electron GK equation is orbit averaged between the mirror reflection points providing a constraint for $g_{\text{el}}$:

$$i(\omega - \langle \omega_{\text{pe}} \rangle)g_{\text{el}} + (C_e(g_{\text{el}})) = (ie(\phi)/T_s)(\langle \omega^2_{\text{pe}} \rangle - \omega)f_{\text{el}},$$

where $\langle \cdots \rangle$ is an average over the bounce-orbit of the trapped electrons. Using WKB-analysis to solve the homogeneous equation and then the method of variation of parameters to determine the solution of the inhomogeneous equation it is possible to construct an approximate solution to the orbit averaged GK equation. The homogeneous solution of the electron gyrokINETIC equation, obtained by WKB-analysis for weakly collisional plasmas, $\dot{\nu} = \nu_e/\omega_0 e^{1/2}, 1$, is

$$g_{\text{hom}}(\kappa) = \frac{1}{(1/2\kappa^2)(c_1 \sinh \kappa^2 + c_2 \cosh \kappa^2)},$$

where $\kappa = \sqrt{2}(\omega_e / \nu_e)^{1/4}$ and $\kappa = [1-\Delta B(1-e)/(2\omega \Delta B)]$. In addition, $B_0$ is the flux-surface averaged magnetic field, $\omega_0 = \omega / \gamma$ is the absolute value of the real part of the eigenfrequency, $\gamma = \sigma + i \dot{\gamma}$, $\sigma = \text{sign}([R(\omega)])$, $\dot{\gamma}$ is the growth rate normalized to $\omega_0$, $u = \dot{\gamma}(2 - \omega_0^2)$, and $\omega_0 = \omega_0 / \omega$ with $\omega_0 = 2k_0^2 \omega/\omega_0$ is the normalized magnetic drift frequency. In the limit $\kappa \to \infty$ (consistent with the assumption $\dot{\nu} \ll 1$), the inhomogeneous part of the distribution can be simplified to

$$g_{\text{inhom}}(\kappa) = \frac{\tilde{S}}{2 \omega(2 - \omega_0)}(4 - \sqrt{\pi \omega^2}),$$

where $\tilde{S} = (e \phi_0 / T_s) / (8/3 \pi) (1+4i/5)(\omega - \omega_0^2)$. In these expressions, we expressed the bounce average of the potential as

$$\langle \phi \rangle = \phi_0 \left[ \frac{E(\kappa)}{K(\kappa)} \right] + \frac{4f_{\text{el}}}{3} \left[ \frac{2(1-\kappa)}{\kappa^2} \frac{E(\kappa)}{K(\kappa)} + 1 - \kappa \right],$$

where $E(\kappa)$ and $K(\kappa)$ are complete elliptic integrals. To obtain Eqs. (5) and (6) we approximated the elliptic integrals with their asymptotic limits for small arguments. Since $g_{\text{el}}(\kappa=0)$ is regular, we choose $c_1=0$, and the boundary condition $g_{\text{el}}(\kappa=1)=0^{1/2}$ gives $c_2=-g_{\text{inhom}}(1)^{1/2} \sinh(2 \nu_e / \nu_e)$ so that the solution for the perturbed trapped-electron distribution is

$$g_{\text{el}} = \frac{\tilde{S}(4 - \sqrt{\pi \omega^2})}{2 \omega(2 - \omega_0)} \frac{\tilde{S}(4 - \sqrt{\pi \omega^2})}{2 \omega(2 - \omega_0)} \frac{\sinh(\kappa^2)}{\sinh(\kappa^2)},$$

except in a narrow layer close to $\kappa=0$ that has a negligible contribution to the velocity space integrals.

The perturbed electron response is proportional to

$$\left\langle \int g_{\text{el}} dv \right\rangle = 4 \sqrt{2e} \int_0^\infty v^2 dv \int_0^1 K(\kappa)g_{\text{el}} d\kappa = \frac{16 \sqrt{2e}}{\omega} \int_0^\infty v^2 dv \tilde{S} \left[ \frac{\tilde{S}}{2 \omega(2 - \omega_0)} + 1 - \frac{\tilde{\nu}_e}{\tilde{E}} \right],$$

where we retained terms only to the lowest order in $\tilde{\nu}_e^{1/2}$ and approximated $\int K(\kappa)g_{\text{el}} d\kappa \approx 2 \int g_{\text{el}} d\kappa$. The expression in Eq. (8) has been compared with a numerical solution to the electron GK equation resulting in excellent agreement for $\tilde{\nu}_e$ up to $O(1)$ values. The velocity integral in Eq. (8) can be evaluated in terms of $\mathbf{F}_0$ generalized hypergeometric functions 18 and the perturbed electron density response becomes

$$\tilde{n}_e/\tilde{n}_e = \frac{e\tilde{\nu}_e}{\tilde{\nu}_e} \left[ \frac{3}{2} \frac{\tilde{\nu}_e}{\tilde{\nu}_e} - \frac{3}{2} \frac{\tilde{\nu}_e}{\tilde{\nu}_e} - \frac{3}{2} \frac{\tilde{\nu}_e}{\tilde{\nu}_e} \right] + \frac{2 \tilde{\nu}_e}{\tilde{\nu}_e} \mathbf{F}_0 \left( \frac{3}{2} \frac{\tilde{\nu}_e}{\tilde{\nu}_e} \right),$$

where $\tilde{\nu}_e = (1+4f_{\text{el}}/5)^2 \omega_0 (3/2) \phi$, $\mathbf{F}_0(z) = 2F_0(\alpha; b; z)$, $\tilde{\nu}_e = \tilde{\nu}_e / \omega_0$, and $\tilde{\nu}_e = \tilde{\nu}_e / \omega_0$. The perturbed electron temperature can be derived from the nonadiabatic electron distribution function given in Eq. (7) to be

$$\tilde{T}_e/\tilde{T}_e = \frac{3 \tilde{\nu}_e}{2} \left( \frac{2 \tilde{\nu}_e}{\tilde{\nu}_e} - \frac{2 \tilde{\nu}_e}{\tilde{\nu}_e} \right) + \frac{7 \tilde{\nu}_e}{4} \mathbf{F}_0 \left( \frac{3}{2} \frac{\tilde{\nu}_e}{\tilde{\nu}_e} \right) \left( \frac{3}{2} \frac{\tilde{\nu}_e}{\tilde{\nu}_e} \right).$$

The hypergeometric functions appearing in the perturbed electron and ion responses can be approximated by a simple algebraic expression $\mathbf{F}_0(z) = (1-\sqrt{b}+y\sqrt{-z})^{-z}$, where the coefficients $c$ and $b$ are given in Appendix B.

B. Ion response

For the ions we neglect the parallel dynamics by assuming $k_B T_i \ll \omega_0$. In this limit Eq. (1) can be solved by neglecting the parallel derivative and replacing $\omega_0$ with its weighted flux-surface averaged value $\langle \omega_{\text{pe}} \rangle$ where $\langle X(\theta) \rangle = \int_0^\pi X(\theta) \psi_{\text{pe}}(\theta)d\theta / \int_0^\pi \psi_{\text{pe}}(\theta)d\theta$. The perturbed ion response becomes

$$\tilde{n}_i/\tilde{n}_i = \frac{e\tilde{\nu}_i}{\tilde{\nu}_i} \left[ \frac{3}{2} \frac{\tilde{\nu}_i}{\tilde{\nu}_i} - \frac{3}{2} \frac{\tilde{\nu}_i}{\tilde{\nu}_i} - \frac{3}{2} \frac{\tilde{\nu}_i}{\tilde{\nu}_i} \right] + \frac{2 \tilde{\nu}_i}{\tilde{\nu}_i} \mathbf{F}_0 \left( \frac{3}{2} \frac{\tilde{\nu}_i}{\tilde{\nu}_i} \right),$$

where $\tilde{\nu}_i = 6 + (9 + 16f_{\text{el}}) \omega_0 / 12(1+i/2)$ and we used the constant energy resonance (CER) approximation for the ion resonance $[\sqrt{v_i^2 + \omega_0^2}]^{-1} \to 4(\sqrt{v_i^2 + \omega_0^2})^{-1/3}$.13
The nonadiabatic part of the ion response can be obtained by the evaluation of the velocity integral of Eq. (11),

$$I_e = \frac{2}{\sqrt{\pi}} \int_0^\infty dx_1 \int_0^\infty dx_2 e^{-x_1^2} \bar{f}_0(x_1/\sqrt{2b_0})$$

$$\times \left[ 1 - \frac{1 - (x_1^2 - 3/2)\eta_i}{1 - x_1^2\bar{a}_{Di}} \right],$$

(12)

where we introduced the FLR-parameter $b_i = b_0(1 + s^2\theta^2)$ with $b_0 = (k_c \rho_i)^2$. Also, $\rho_i = \sqrt{\eta_i/r_i}$ is the ion sound Larmor radius, $\tau = T_e/T_i$ is the electron-to-ion temperature ratio, and $c_i = \sqrt{T_i/m_i}$ is the ion sound speed. In order to make further progress analytically, we restrict our analysis to long wavelength perturbations and keep only the linear terms in $b_0$. This approximation is typically valid for the fastest growing ITG modes ($k_c \rho_i \sim 0.2$). Then $I_e$ can be evaluated to obtain the perturbed ion response

$$\frac{\dot{n}_i}{n_i} \frac{e\phi}{T_i} = -\bar{a}_{ni} + \left( \frac{3\bar{a}_{Di}}{2} - b \right)$$

$$\times \left[ \omega_{qi} \frac{5}{2} \eta_i \bar{a}_{qi} - \bar{a}_{Di} \omega_{qi} \Sigma_{7/2}^i(\bar{a}_{Di}) \right],$$

(13)

where $b = (b_0, \frac{\eta_i}{r_i} = b_0[1 + s^2(2\pi^2 - 12 + 12i)(2\pi^2 - 3)]/\sqrt{(1 + i\beta_i)}$ is the weighted flux-surface averaged value of the FLR parameter. Note that the expressions for the perturbed electron and ion density in Eqs. (9) and (13) are exact in $\omega_{qi}$; no approximation regarding the relative magnitude of $\omega_{qi}$ and $\omega$ has been made. Evaluating an integral similar to Eq. (12) but with the integrand multiplied by $x_1^2$ leads to the nonadiabatic perturbed ion temperature response,

$$\frac{\dot{T}_i}{T_i} \frac{e\phi}{T_i} = \frac{3}{2} \left[ 1 - (1 + \eta_i)\bar{a}_{ni} \right] + \frac{5}{2} \left( \frac{3\bar{a}_{Di}}{2} - b \right)$$

$$\times \left[ \omega_{qi} - \frac{7}{2} \eta_i \bar{a}_{qi} - \bar{a}_{Di} \omega_{qi} \Sigma_{7/2}^i(\bar{a}_{Di}) \right].$$

(14)

C. Nonresonant expansion

In the limit of low normalized magnetic drift frequencies, expanding Eq. (9) around $\omega_{Di} = 0$ and keeping only the first order terms (usually called the nonresonant expansion), the perturbed electron density reduces to the following expression:

$$\frac{\dot{n}_e}{n_e} \frac{e\phi}{T_e} = 1 - \bar{a}_e$$

$$\sqrt{2e} \left[ 1 - \bar{a}_e + \frac{3\bar{a}_{De}}{4} \left[ 1 - (1 + \eta_e)\bar{a}_{ve} \right] \right]$$

$$- \frac{2\Gamma(\frac{3}{2}) \bar{e}_v \bar{e}_i}{\sqrt{\pi} \eta_i} \left[ 1 - \left( 1 - \frac{3\eta_e}{4} \right) \bar{a}_{ve} \right]$$

$$+ \frac{9\bar{a}_{De}}{16} \left[ 1 - \left( 1 + \frac{\eta_e}{4} \right) \bar{a}_{ve} \right].$$

(15)

For the ions, expanding Eq. (13) in $\omega_{Di}$ leads to

$$\frac{\dot{n}_i}{n_i} \frac{e\phi}{T_i} = -\bar{a}_{ni} + b - [1 - b(1 + \eta_i)]\bar{a}_{ni}$$

$$+ \left[ (3 - 5b) - [3(1 + \eta_i) - 5b(1 + 2\eta_i)]\bar{a}_{ni} \right] \frac{\bar{a}_{Di}}{2}.$$  

(16)

However, as we will show, the results based on the expansions in $\omega_{Di}$ and $\omega_{Di}$ in Eqs. (15) and (16), respectively, will give large errors compared to the exact solutions in Eqs. (9) and (13). The inability of the nonresonant expansion to reproduce the correct eigenvalues and fluxes has been noted before in Ref. 12. The reason for this is illustrated in Fig. 2 where the difference between the exact and an expanded solution is shown for one of the hypergeometric functions. The solid lines correspond to the real (black) and imaginary (blue) parts of the generalized hypergeometric function appearing in the ion response [Eq. (13)], their expansion to first order around $\omega_{Di} = 0$ are plotted with dotted lines, while the dashed lines correspond to a simple algebraic approximation discussed in Appendix B. The ITG mode with $\eta = 0.5$, $\omega_0 = -\omega_{ve}$ for $R/L_i = 3$, $s = 1$ would correspond to $\omega_{Di}/2 = 0.84$.

D. Summary of analytical formulas

Using the approximative formula for the hypergeometric functions from Appendix B, the perturbed electron density, and temperature responses, $\dot{n}_e = n_i T_e/(n_e e \phi)$ and $\dot{T}_i = T_e/\epsilon \phi$ can be written as

$$\dot{n}_e = 1 - \bar{a}_e$$

$$\sqrt{2e} \left[ \omega_{qi} - \frac{3}{2} \left( \eta_i \omega_{qi} - \omega_{Di} \omega_{qi} \right) \right]$$

$$- \frac{2\Gamma(\frac{3}{2}) \bar{e}_v \bar{e}_i}{\sqrt{\pi} \eta_i} \left[ 1 - \left( 1 - \frac{3\eta_e}{4} \right) \bar{a}_{qi} \right]$$

$$+ \frac{3\eta_i \bar{a}_{qi}}{16} \left[ 1 - \left( 1 + \frac{\eta_e}{4} \right) \bar{a}_{qi} \right].$$

(17)
The perturbed electron density and temperature responses are
\[ \tilde{n}_e = \tilde{n}_i + \left( \frac{3\tilde{\omega}_{Di}}{2} - b \right) \times \left[ \tilde{\omega}_{pe} - \frac{7(\eta \tilde{\omega}_{si} - \tilde{\omega}_{Di} \tilde{\omega}_{pe})}{2(2/3 + \sqrt{3}/9 - \tilde{\omega}_{pe})} \right], \]
\[ \tilde{T}_e = \frac{3}{2} \sqrt{2} \left[ -\tilde{\omega}_{si} + \left( \frac{3\tilde{\omega}_{Di}}{2} - b \right) \right] \times \left[ \tilde{\omega}_{pe} - \frac{5(\eta \tilde{\omega}_{si} - \tilde{\omega}_{Di} \tilde{\omega}_{pe})}{2(9/14 + \sqrt{5/14} - \tilde{\omega}_{Di}/2)^{5/2}} \right]. \]

The formulas, which we summarized above, are the most accurate known for moderate magnetic shear and they can be used to compute the dispersion relation and the quasi-linear fluxes as shown in the following chapters. They are useful in showing the scalings with collisionality, magnetic drift frequency, diamagnetic direction and ratio of the density and temperature scale lengths.

### III. STABILITY

The dispersion relation follows from the quasineutrality condition \( \tilde{n}_i = \tilde{n}_e \), where the perturbed electron and ion densities are given by Eqs. (9) and (13), respectively, and we take a flux-surface average. The dispersion relation obtained here is valid for both ITG propagating in the ion diamagnetic direction (\( \sigma = -1 \)) and TE modes propagating in the electron diamagnetic direction (\( \sigma = 1 \)), but in this paper, we will focus only on the ITG mode stability and the quasi-linear fluxes driven by them. The effect of collisions modeled by a Lorentz operator on the stability of TE modes has been studied before in the steep density and temperature gradient region, where the curvature drift can be neglected.

In the limit of large aspect ratio, \( \epsilon \rightarrow 0 \), the trapped part of the perturbed electron density can be neglected and the dispersion relation reduces to the following expression for ITG (\( \sigma = -1 \)) modes with adiabatic electrons:

\[ 1 = \tau \left[ -\tilde{\omega}_{si} + \left( \frac{3\tilde{\omega}_{Di}}{2} - b \right) \right] \times \left[ \tilde{\omega}_{pe} - \frac{5(\eta \tilde{\omega}_{si} - \tilde{\omega}_{Di} \tilde{\omega}_{pe})}{2(9/14 + \sqrt{5/14} - \tilde{\omega}_{Di}/2)^{5/2}} \right], \]

Using the condition of marginal instability \( \gamma = 0 \), we can derive an approximate stability condition for the ITG modes. We start by noting that the imaginary part of \( \tilde{\omega}_{pe} \) is negligible if \( (7s - 6)f_i/(6 + 9s + 16f_i^2) \ll 1 \) and the imaginary part of \( b \) is also negligible when \( 9f_i s^2/[2(3 + (\pi^2 - 6)s)] \ll 1 \) and \( f_i^2 \ll 1 \), in which case the expression in Eq. (21) is real except for the term containing the function \( F_{2/3}(\tilde{\omega}_{Di}) \) that has an imaginary part for all values except \( \tilde{\omega}_{Di} = 0 \). Therefore, the condition \( \gamma = 0 \) can only be satisfied if the coefficient of \( F_{2/3}(\tilde{\omega}_{Di}) \) vanishes. Using \( \tilde{\omega}_{si} = -\tilde{\omega}_{pe} \tau \) and \( \tilde{\omega}_{Di} = -2\tilde{\omega}_{pe} \epsilon_1/(\tau) \), where \( \epsilon_1 = L_n/R \), the \( \omega_{pe} \) for which the coefficient of \( F_{2/3}(\tilde{\omega}_{Di}) \) is zero can be shown to be

\[ \omega_{pe} = \frac{(2 + 3s)(3 \eta - 2)\epsilon_1 \omega_{ve}}{2f_i(2 + 3s)\epsilon_1 - 3 \eta \epsilon_1}, \]

where we set \( \sigma = -1 \) for ITG modes. The critical \( \eta \) for stability satisfies the remaining part of Eq. (21),

\[ \frac{1}{\tau} = \frac{3\tilde{\omega}_{Di}}{2} - b \tilde{\omega}_{pe} = \tilde{\omega}_{si}. \]

Equation (23) can be rewritten as \( 3(b-1)\eta \tau + (2+3s)(1 + \tau)\epsilon_1 = 0 \), so that the ITG stability boundary for adiabatic electrons becomes

\[ \eta_{\text{crit}} = \frac{1}{\tau} \left( 1 + \frac{1}{\tau} (2 + 3s)\epsilon_1 \right), \]

and the corresponding critical real frequency of the mode is

\[ \frac{\omega_{pe}}{\omega_{ve}} = \frac{b - 1}{\tau b + 1} + \left( 1 + \frac{1}{\tau} (2 + 3s)\epsilon_1 \right). \]

For \( b = 0 \) and \( s = 1 \), the critical \( \eta \) given in Eq. (24) is similar to what was found previously in the local kinetic limit: \( \eta_{\text{crit}} = (1 + 1/\tau)\epsilon_1/3 \). In the present model the coefficient of \( (1 + 1/\tau)\epsilon_1 \) is \( 5/3 \) for \( s = 1 \), because the flux surface average of the magnetic drift frequency was used instead of its value at \( \theta = 0 \) as in Ref. 13.

If we retain the trapped electron contribution, in the limit of low collisionality \( \nu_e/(\omega_{te}) \ll 1 \) the dispersion relation becomes
energy flux for particle species
\(a\) is defined by
\[
\bar{\tau}_e = \frac{k_B\phi_a}{eB} \bar{T}_e, \quad \bar{\tau}_i = \frac{k_B\phi_a}{eB} \bar{T}_i
\]
where the overbar denotes the flux-surface average of the perturbed quantities and \(\bar{\phi} = \phi_0(1 + \text{if}_p)/2\). The quasilinear energy flux for particle species \(a\) is defined by
\[
Q_a = -k_B\phi_a T_a \frac{\bar{\phi}}{eB} \left( \frac{\bar{T}_e}{T_e} \right) = \frac{\bar{\phi}}{eB} \left( \frac{\bar{T}_e}{T_e} \right) \frac{\bar{n}_e}{n_e}
\]
The quasilinear fluxes can be evaluated using the expressions for the perturbed electron and ion density and temperature responses given in Sec. II in Eqs. (9), (11), and (14). Quite accurate approximate results can be obtained also from the formulas listed in Sec. II C, Eqs. (17), (19), and (20), respectively.

In the following we will present the \(a/l_{T_i}\) collisionality, and shear scalings of the eigenfrequency, growth rate, particle flux and electron and ion energy fluxes, together with quasilinear and nonlinear GYRO results for the following parameter set \(a/l_{T_i} = a/l_{T_i} = 3, \quad x = 1, \quad q = 2, \quad a/R = 1/3, \quad a/R = 2, \quad a/l_{T_o} = 1\) (i.e., the GA standard case). Each GYRO nonlinear simulation used a perpendicular domain size of \((L_x/l_p, L_y/l_p) = (86, 90)\), fully resolving modes with wave numbers in the range \(k_y l_y = 2.3\) and \(k_y l_y = 1.1\). The standard 128-point velocity-space grid (eight energies, eight pitch angles, and two signs of velocity) was used. Electrons were taken to be drift kinetic with \(\nu_{\text{ne}}/m_e = 60\) and the simplified \(s = \alpha\) geometry equilibrium model was used.

A. \(a/l_{T_i}\) scaling

Figure 4 shows the \(a/l_{T_i}\) scalings of the eigenvalues and the fluxes in the collisionless case. The COMET results (solid line) are compared with quasilinear GYRO (blue dash-dotted line) and nonlinear GYRO (red dots) simulation results. For comparison, the results computed by expanding to first order in \(\alpha_{T_i}/\omega\) (nonresonant expansion) are shown with green dashed lines and the results using the algebraic approximations to the hypergeometric functions are shown with purple dotted lines. The frequencies are normalized to \(c_s/a\), where \(a\) is the minor radius, while the particle and energy fluxes are normalized to \(k_B\phi_a/eB \bar{T}_e\) and \(k_B\phi_a T_a eB \bar{\phi} / T_a^2\), respectively. The comparisons to the nonlinear simulations are based on the choice of the flux surface averaged perturbed potential amplitude \(\bar{\phi}\) so that \(eB \bar{T}_e / \rho_s = 6.5\), where \(\rho_s = \rho_p / \rho\), which is consistent with the usual mixing length estimate \(\bar{n}_e / n_e \sim 1/ (k_B \mu_m)\). The agreement between COMET and quasi-linear GYRO results is very good. The disagreement with the nonresonant expansion (green dashed line) is large, but not surprising, since \(\alpha_{T_i} \sim \omega\).
The particle fluxes are inward and their absolute values decrease with $a/L_{T_i}$, but the electron and ion energy fluxes increase with $a/L_{T_i}$. The disagreement between the full COMET solution and the nonresonant expansion is remarkable for the particle flux, which shows that the nonresonant expansion fails to reproduce both the sign and the magnitude of the particle fluxes. Furthermore, the nonresonant expansion gives incorrect scaling for the electron energy flux as a function of $\eta$.

B. Collisionality scaling

Figure 5 shows the collisionality scaling of the eigenvalues and fluxes. As in Fig. 4 we show the COMET results compared to quasilinear and nonlinear GYRO results together with the nonresonant expansion. The collisionality $\nu_{c,i}$ is defined in units of $c_e/\alpha$ and is $\nu_{c,i} \equiv (n_i e^4 \ln(A)/[4\pi e^2 (2T_e)^{1/2} m_i])$, where $\ln(A)$ is the Coulomb logarithm. The results show that the eigenfrequency and growth rate of the ITG mode are insensitive to the collisionality, therefore the quasilinear particle flux driven by ITG is almost identical to the one calculated in Ref. 11, where the collisionality dependence of the eigenvalues was neglected. As noted before, the particle flux changes sign from inward to outward at a certain value of the collisionality. The electron and ion energy fluxes depend only weakly on collisions. Again, the agreement between COMET and GYRO results is very good. Also here, the nonresonant expansion gives incorrect scaling for the electron energy flux as a function of $\eta$.

C. Shear scaling

The validity of the model depends on the assumed electrostatic potential, the form of which is dependent on the shear. Some of the shear dependence is kept by using a shear-dependent factor $f_s$ multiplying the imaginary part of the potential, but the width of the real part of the potential is not varied and the dependence of $f_s$ on plasma parameters other than shear is neglected. Therefore the model is still too crude to capture all of the shear dependence of the problem. However, as illustrated in Fig. 6, where the shear dependence of the eigenvalues and fluxes are shown in the collisionless case, in the moderate shear region, the COMET results have reasonable agreement with quasilinear GYRO results.

V. CONCLUSIONS

In this paper we presented a semianalytical collisional model for electrostatic microinstabilities and the quasilinear transport fluxes driven by them. By assuming a ballooning eigenfunction for the electrostatic potential we obtained closed analytical expressions for the perturbed electron/ion density and temperature responses. The expressions contain
explicitly the dependence on electron-ion collision frequencies and no expansion in the smallness of \( \omega_D/\omega \) is used. The collisions are modeled by the Lorentz operator, which gives the proper boundary layer development of the nonadiabatic trapped electron distribution at the trapped-passing boundary as it is shown in Ref. 11. We illustrated that the linear approximation of the hypergeometric functions, the so-called nonresonant expansion, is valid only in a very small vicinity of \( \omega_D/\omega \approx 0 \). Therefore it is not appropriate for typical ITG frequencies and it gives incorrect results for the parameter region we studied, where the density profile is not too steep. We introduced a simple algebraic approximation of these functions for \( \omega_D/\omega = O(1) \) arguments.

Our model is semianalytical in the sense that the roots of the dispersion relation are obtained numerically. However, we found an exact ITG stability boundary in Eq. (24) for the \( \epsilon \rightarrow 0 \) and \( \omega_D/\omega \rightarrow 0 \) limits, which incorporates not only the \( L_T/R \) and \( T_i/T_e \), but also the shear and the FLR-parameter dependences. This stability boundary agrees well with numerical results obtained in GYRO. It has been shown that for a constant density gradient, increasing collisionality leads to a larger \( a/L_T \) threshold for stability. This means that decreasing collisionality gives rise to a destabilization of ITG mode driven turbulence, which in turn may lead to a steepening of the density profile.

We benchmarked our model to the recognized, state-of-the-art gyrokinetic simulation code GYRO and illustrated the agreement on the GA standard case.\(^5\) The results for the quasilinear particle flux changes dramatically for very low collisionality, in agreement with the results of Ref. 11. We illustrated that the linear approximation of these collisions restrict the validity of the model, it is still instructive to study what determines the shape of the ballooning potential.

The frequencies and growth rates agree well with quasilinear gyrokinetic calculations with GYRO. Our results show that the toroidal ITG frequencies can be calculated accurately neglecting the parallel ion dynamics using a model ballooning potential. However it is important to take the shear dependent imaginary part of the potential into account. We motivated the model potential by a self-consistent variational solution of the ballooning eigenfunction problem.

The frequencies and growth rates of the ITG modes far from threshold are not very sensitive to the collisionality, but the quasilinear particle flux changes dramatically for very small collisionalities, in agreement with the results of Ref. 11 and of previously published gyrokinetic simulations, e.g., in Ref. 12. Expressions for the electron and ion particle and energy fluxes have been derived and from comparisons with quasilinear and nonlinear GYRO simulations we concluded that the COMET flux captures the qualitative collisionality dependence and \( a/L_T \)-scalings. The quantitative agreement between COMET and GYRO becomes worse for low and large shear, because the assumed form for the electrostatic potential is not a good approximation in that region. Reliable quantitative predictions for the eigenvalues and fluxes can therefore only be obtained in the moderate shear region.

**ACKNOWLEDGMENTS**

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**APPENDIX A: MODEL FOR THE PERTURBED ELECTROSTATIC POTENTIAL**

In order to motivate the chosen model for the perturbed electrostatic potential, it is convenient to follow a variational approach using a trial function for the perturbed electrostatic potential motivated by GYRO simulations:

\[
\phi(\theta) = \phi_0[1 + A \cos \theta + B \sin^2 \theta + (1 - A)\cos 2\theta] \\
\times [H(\theta + \pi) - H(\theta - \pi)].
\] (A1)

The coefficients \( A \) and \( B \) are determined by taking \( \{1, \cos \theta, \sin^2 \theta, \theta\} \) moments of the integrodifferential equation derived from the quasineutrality condition. A similar method was used by Ref. 19, where a trapped particle instability was considered. The potential was expanded in series in \( \cos \theta \) and the coefficients were determined by taking moments of the equation resulting from the quasineutrality condition. More recent attempts to obtain a model electrostatic potential (see, e.g., Ref. 14) neglected the effect of the trapped particles, motivated by the smallness of the inverse aspect ratio. The neglect of the trapped population simplifies the mathematical problem considerably, since it removes the integral part of the equation. In reality the factor \( \sqrt{2} \) is not very small and the trapped population may have influence on the form of the potential. With the variational method outlined by Ref. 19, the effect of the trapped particle population can be kept, along with the terms resulting from the ion parallel motion. Unfortunately a full analytical solution of the problem is difficult, unless we neglect collisions and expand in the smallness of \( \nu_T/\omega_D \), the FLR parameter \( b \) and normalized magnetic drift frequency \( \omega_D/\omega \). Although these approximations restrict the validity of the model, it is still instructive to study what determines the shape of the ballooning potential.

We begin with a simplified version of the problem, by neglecting the \( \omega_D \)-resonance, ion parallel motion and collisions. Then, the ion response can be obtained by neglecting the term proportional to \( \omega_D \) in Eq. (16) and is

\[
\frac{\ddot{\tilde{n}}_i}{en_i/T_e} = -\{\tau\tilde{\omega}_i + \tau b_i(1 - \tilde{\omega}_s)(1 + \eta_i)\} \phi,
\] (A2)

where \( b_i = b_\lambda(1 + \hat{s}^2 \hat{\rho}^2) \) and the tilde denotes normalization with respect to \( \omega \). Neglecting collisions, the electron response can be obtained directly from Eq. (4),

\[
\frac{\dot{n}_e}{en_e/T_e} = \phi - (1 - \tilde{\omega}_s)\frac{1}{2}\frac{B\lambda}{\sqrt{1 - \lambda B}} \phi,
\] (A3)

where \( \langle \cdots \rangle \) is an average over the bounce-orbit of the trapped electrons and the \( \lambda \)-integral is over the trapped electron population. Using Eqs. (A2) and (A3) the quasineutrality condition becomes

\[
\text{phys. plasmas} 16, 072305 (2009)
\]
0 = \frac{\hat{n}_i - \hat{n}_i}{en_s(1 - \omega_n)} \\
= \frac{1 + \tau \hat{\omega}_i + \tau b_0 (1 - \omega_n)(1 + \eta)}{1 - \omega_n} \frac{1}{2} \int \frac{Bd\lambda}{\sqrt{1 - \lambda B}} (\phi) , \hspace{1cm} (A4)

and it can be written as an integral equation

\[ [q_0 + (q_0 - 1) \tau^2 \theta^2] \phi - \frac{1}{2} \int \frac{Bd\lambda}{\sqrt{1 - \lambda B}} (\phi) = 0 , \hspace{1cm} (A5) \]

where

\[ q_0 = \frac{1 + \tau \hat{\omega}_i + \tau b_0 (1 - \omega_n)(1 + \eta)}{1 - \omega_n} \hspace{1cm} (A6) \]

For the electrostatic potential we assume the trial function given in Eq. (A1) and take moments of the quasineutrality equation,

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{\hat{n}_e - \hat{n}_i}{en_s(1 - \omega_n)T_e} \left[ \frac{1}{\cos \theta} \sin^2 \theta \right] = 0, \hspace{1cm} (A7) \]

to obtain three equations that determine the coefficients of the perturbed potential \( A, B \), and the eigenvalue \( q_0 \) that gives the dispersion relation.

For the parameters used throughout this paper (\( \tau = 1 \), \( \eta = \eta_s = 3 \), \( s = 1 \), \( q = 2 \), and \( L_n/R = 1/3 \)), the above simplified approach gives the eigenvalue \( q_0 = 0.46 \) and the coefficients for the eigenfunction: \( A = 0.75 \) and \( B = -1.3 \). The real part of the potential is similar to the one obtained by gyro, but due to the neglect of the parallel ion motion and \( \omega_D \) resonance, the information about the imaginary part of the potential is lost. Note that for the simple ion response given in Eq. (A2) the parameters \( A, B \), and \( q_0 \) can be calculated without any assumption on the mode frequency.

Including parallel ion motion and the effect of the \( \omega_D \) resonance to leading order, the ion response is given by

\[ \frac{\hat{n}_i}{en_s/T_e} = \phi \int \left[ -\hat{\omega}_i - b_0 (1 - \omega_n)(1 + \eta) \right] 
+ \frac{q_0 - 1}{2} (1 - \omega_n) \left[ - b_0 \phi^2 + \frac{3 \hat{b}_0}{2} \right] 
+ \frac{\phi''}{\phi} \left[ 5 b_0 - \frac{1}{2} + \frac{5}{3} \left( b_0 \phi^2 - \frac{15 \hat{b}_0^2}{4} \right) \right] \right] \right] \right] , \hspace{1cm} (A8) \]

where the term proportional to \( \hat{s} = v_T^2 / \omega^2 q^2 R^2 \) represents the contribution from the parallel ion motion and we used the CER approximation. Neglecting electron-ion collisions and the \( \omega_D \) resonance the electron response is given by Eq. (A3).

The shape of the ballooning potential and the mode frequency \( \omega \) can be self-consistently calculated using the following iteration scheme. By the solution of the three coupled equations resulting from the integrals given in Eq. (A5), the parameters \( A, B \), and \( q_0 \) can be determined for a given \( \omega \). A new \( \omega \) can be obtained from the solution of Eq. (A4), which is fed back to the previous system of equations. For our standard parameters it leads to the eigenvalue: \( q_0 = 0.99 \pm 0.11i \), mode frequency \( \omega = \omega_n (1.0 \pm 2.3i) \), and the coefficients for the perturbed potential are \( A = 1.1 \pm 0.060i \) and \( B = 0.049 \pm 0.59i \). The potential corresponding to these values is plotted in Fig. 1 with green dashed line, together with the one computed by gyro with and without ion parallel motion, and the one we use in the paper. The potential calculated in this way agrees very well with the result of the gyro simulation. We note that taking into account the effect of the \( \omega_D \)-resonance to first order and parallel ion motion is important to obtain the correct sign of the factor multiplying the imaginary part of the potential.

APPENDIX B: APPROXIMATION OF THE HYPERGEOMETRIC FUNCTIONS IN THE RELEVANT PARAMETER REGIME

Since the frequency of the toroidal ITG mode is typically of the order of the magnetic drift frequency, the nonresonant expansion, i.e., the expansion in the smallness of \( \omega_D / \omega \), is not appropriate, as it is illustrated on Fig. 2. In order to obtain simple algebraic expressions for the perturbed density and temperature responses [given in Eqs. (9), (13), (11), and (14), respectively] we introduce approximations for the generalized hypergeometric functions \( F(a; z) \) appearing in these expressions. These approximations are valid for \( |z| = O(1) \) arguments and exact in the \( z \rightarrow 0 \) limit. In our analysis, we assumed that \( \omega_D \ll \omega \), which gives an upper limit for the argument of the hypergeometric functions, namely, \( z \sim \omega_D / \omega \) cannot be much higher than one. Therefore we do not needed the approximation to be valid asymptotically as \( z \rightarrow \infty \).

For arguments that are not too large (\( |z| < 3 \)), we can approximate the generalized hypergeometric function \( F(a; z) = F(a; z) \) by

\[ F(a; z) \approx (1 - b + 2b - z)^c \hspace{1cm} (B1) \]

where \( b = c^2 / (4a_1 a_2) \) and \( c \) is a constant that can be found by the minimization of the absolute value of the integral difference between the exact and approximate functions for a finite radius domain of the complex plane. The constant is not a low order rational number in general, but can be approximated with a properly chosen one, still providing a very good approximation for \( F(a; z) \). The values of \( c \) corresponding to the best approximation in the above sense for \( |z| < 3 \) and their rational approximations are listed in Table I. The average integral difference of the approximative algebraic expression and the exact hypergeometric function on the \( |z| < 3 \) domain was shown to be below 4% if the constant \( c \) is chosen to be a rational number. In Figs. 4–6 the eigenvalues and

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<th>( c )</th>
<th>( c^\infty )</th>
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<tr>
<td>1.0</td>
<td>2.56</td>
<td>2.89</td>
<td>1.21</td>
<td>2.35</td>
<td>3.17</td>
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fluxes calculated using the approximative formula (B1) with the rational approximations of \( c \) are shown with purple dotted lines. The eigenvalues and fluxes calculated with the approximation in Eq. (B1) are in excellent agreement with the ones calculated with the exact hypergeometric functions.

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Paper C

Deconvolution-based correction of alkali beam emission spectroscopy density profile measurements

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A deconvolution-based correction method of the beam emission spectroscopy (BES) density profile measurement is demonstrated by its application to simulated measurements of the COMPASS and TEXTOR tokamaks. If the line of sight is far from tangential to the flux surfaces, and the beam is one-dimensional and neglecting its finite width, the properties of the measurement in account in terms of the fluctuation’s time scale in order to obtain a complex picture of the cross field turbulent transport.7 We have to point out that the high temporal resolution fluctuation measurements need always to be combined with the significantly slower profile measurement.

The collisional-radiative model is considered in the standard description of beam evolution.10 Based on this model, reliable numerical methods were developed to determine the density profiles along the beam line, given the observed light profile,7,11,12 such as the Li-BES density reconstruction code ABSOLUT, which is used in the present study.7 All these methods are based on the assumption that the measured light profile is equivalent to the emissivity of the beam. This case corresponds to the ideal measurement geometry considering the beam to be one-dimensional (1D) and neglecting its finite width.

Our goal is to provide a tool to quantitatively measure the systematic error caused by this simplified treatment. Partially for this purpose, as well as to support the design and interpretation of BES measurements, the RENATE alkali BES simulation code has been developed. For the purpose of this study, alkali BES setup and plasma parameters are chosen from the recently upgraded TEXTOR Li-BES diagnostic,13 and the alkali BES diagnostic planned for the newly restarted COMPASS.14 We investigate the character and magnitude of the systematic error in density profile reconstruction, and conclude that it can be significant in certain, experimentally relevant cases, which we support with a general estimation of the maximal error in the calculated electron density. The simulation of the phenomenon also enabled the design of a method for the correction of the measured light profile re-

I. INTRODUCTION

Lithium beam emission spectroscopy (Li-BES) is an active diagnostic tool, typically probing the outer regions of fusion plasmas by observing the characteristic emission of a 10–100 keV atomic beam injected into the plasma.1,2 Li-BES measurements are routinely performed on several fusion devices.3 Application of sodium for BES purposes is recently considered,4,5 thus we refer to the method as alkali BES hereafter.

The evolution of the populations of different atomic states depends on the distribution of plasma parameters along the beam line. This means that from the emitted intensity distribution at a characteristic frequency (i.e., the light profile) information can be obtained on the distribution of electron density6 and its fluctuation.2

Advantages of the alkali BES diagnostic include being practically nonintrusive—because of the low density of the beam—and being approximately a point measurement. The technique is exceedingly suitable for scrape-off layer (SOL) and pedestal density profile measurements with good temporal and spatial resolution (50 μs, 5 mm),7 thus contributes to the understanding of transitions between different confinement modes and to the validation of models of edge transport barrier formation.8 Statistical behavior of density fluctuations with approximately microseconds time scale in the edge and SOL (e.g., radial wave number spectra, correlation length and time) can also be investigated in the radial direction by “single beam” fluctuation measurements.9 Moreover, fluctuation measurements can be extended to two dimensions by electrostatically deflecting the beam in the poloidal plane in
ducing the effect of the finite beam width, and thus allowing the use of the 1D density reconstruction methods.

The structure of the paper is as follows: In Sec. II the alkali BES measurement simulation, RENATE is introduced. The issues of the observation of a finite width beam are investigated in Sec. III. The emission reconstruction correction method is discussed in Sec. IV, and demonstrated through realistic simulated measurements in Sec. V. A general estimate of the error due to finite beam width is given in Sec. VI, and finally, the results are summarized in Sec. VII.

II. RENATE ALKALI BES MEASUREMENT SIMULATION

For the purpose of supporting the design of alkali BES density profile and fluctuation measurements, an Interactive Data Language (IDL) simulation code, RENATE, has been developed which also assists the interpretation and correction of measured data. In order to take the finite width of the beam into account, the beam evolution is calculated separately in slices of the beam considering a realistic current distribution. The integration of emitted light along the lines of sight is modeled together with other essential features of the observation and the detector system.

The atomic physics processes of the beam are modeled by the collisional-radiative model. The rate equations describing the evolution of atomic occupations can be written in a quite compact form

\[ \frac{dn_i}{dx} = \sum_j [n_i(x) a_{ij}(x) + b_{ij}] n_j(x) \quad (i, j = 1, \ldots, m), \]

where \( n_i \) is the electron density, \( n_i \) and \( n_j \) are the populations of the \( i \)-th and \( j \)-th atomic states, respectively, and \( x \) is the coordinate along the beam. The atomic transition and electron loss processes due to electron (e), proton (p), and impurity (I) collisions are described by the reduced rate coefficients \( a_{ij} \).

Taking the effect of the impurities into account through one representative impurity characterized by charge \( q(x) \) and producing an effective ion charge \( Z_{eff}(x) \), the matrix is written as \( a_{ij} = a_{i'j'} + (1 - qf) a_{i'j} + af \), where \( f = (Z_{eff} - 1)/[q(q-1)] \). The spontaneous atomic transitions are described by the \( b_{ij} \) matrix. In the simulation, the numbers of registered atomic levels are \( m = 9 \) for lithium and \( m = 7 \) for sodium. Note that \( a_{ij} \) depends on \( x \) not only through \( q \) and \( Z_{eff} \) but due to the temperature dependence of the rate coefficients. The ion and impurity temperatures are chosen to be equal to the electron temperature, causing only a negligibly small error, due to the flat temperature dependence of \( a_{ij} \) and \( a_{ij} \).

The photon emission density of a beam per unit time and length is proportional to \( n_i I A_x e_B \), where the observed spectroscopic line corresponds to the \( \lambda \rightarrow \lambda \) transition (2p \( \rightarrow \) 2s for Li, 3p \( \rightarrow \) 3s for Na), \( e_B \) is the beam velocity, \( A_x \) is the corresponding Einstein coefficient, and \( I \) is the beam current. The \( n_i \) population is calculated by the solution of the direct problem, integrating Eq. (1) stepwise from the point where the beam enters the plasma \( x = 0 \), with the initial condition \( n_i(0) = \delta_{i1} \), where \( 1 \) is the index of the ground state.

Since the RENATE code was originally developed with the purpose of design of BES measurements, it solves the direct problem calculating the beam evolution and the emission distribution for a given measurement configuration and set of plasma parameters. Therefore, the most important component of the simulation is its atomic physics kernel, which calculates the rate coefficients from parametrically given cross sections of the collisional processes and solves the rate of Eq. (1) by a fourth order Runge–Kutta method. The spontaneous atomic transition probabilities are taken from the National Institute of Standards and Technology (NIST) atomic spectra database, and the cross section data found in Refs. 15–17 for lithium and Ref. 4 for sodium are used. The collisional \( j \rightarrow i \) de-excitation rate coefficients are derived from the corresponding \( i \rightarrow j \) excitation rate coefficients using the principle of detailed balance, while impurity collision rate coefficients are calculated from the proton collisional cross sections using the scaling relations given in Refs. 4 and 15. The proton impact target electron loss processes are considered instead of treating the ionization and charge exchange channels separately, which is the main difference of the atomic physics kernel from the ABSOLUT (Ref. 6) inverse problem solver regarding the atomic physics.

ABSOLUT has a corresponding direct solver code called SIMULA, which we used for the validation of RENATE. The found relative difference between the rate coefficients calculated by the different programs is \( O(10^{-9}) \), and accordingly the maximum relative difference between the calculated evolution of atomic populations is the same order of magnitude. The ABSOLUT code in turn has been critically tested against both Li (Ref. 6) and Na (Ref. 5) measurements. In this manner, RENATE is indirectly validated to measurements; the direct validation is under way at the TEXTOR tokamak.

The calculation scheme of the simulation is as follows. First, the beam is divided into slices which are perpendicular to the poloidal plane while the velocity of the beam atoms is tangential to them. The emissivity profile is calculated along each slice, given the magnetic geometry \( \Psi(R, Z) \), together with the distribution of the relevant plasma parameters, \( n_e, T_e, q, \) and \( Z_{eff} \) as a function of a flux coordinate \( \Psi \). The plasma parameters are assumed to be equally distributed on a flux surface. Then the calculation of the geometric efficiency (i.e., the effect of that the collecting optical element covers different solid angles seen from different points of the beam) is performed for the points of each slice, and the efficiency of the observation system is taken into account in order to determine the number of photons per unit time detected by each detector segment. Contributions of the different beam slices and points to the detected signal are summed up.

We restrict our studies to measurement geometries where the beam axis is in the poloidal plane and the “observation point” is also located in the same toroidal position, which is typical for diagnostic neutral beams. In this case, we can project the three-dimensional beam into the poloidal plane of the beam axis, and the observed volumes reduce to observed areas.

III. OBSERVATION OF A FINITE WIDTH BEAM

The observed signal from a diagnostic beam is equal to the integral, along the line of sight, of the emissivity...
weighted by the geometric efficiency. Since the beam has a finite width, a line of sight goes through parts of the beam being in different stages of beam evolution, thus the measurement cannot be perfectly local. Inverting this effect, the emission reconstruction method gives an estimate of the emissivity on the beam axis from a measured light profile.

We denote the coordinate measured along the beam axis by \( x \), and index each segment of the detector array by \( x' \), marking the position where the middle of the observed volume of the detector segment intersects the beam axis (see Fig. 1). For the sake of simplicity of the formalism, \( x' \) is also considered to be a continuous independent variable.

Assuming that the plasma parameters are flux functions, it can be concluded from our simulations that the evolution of atomic populations also follows the flux surfaces, except from extreme cases of wide beams injected almost tangentially to the flux surfaces. This enables us to extend the emissivity along the beam axis \( I(x) \) into two dimensions by mapping along the flux surfaces indexed by \( x'' \) marking their intersection with the beam axis and weighting with the beam current distribution. Thus, we can express the measured light profile \( S(x') \) as

\[
S(x') = \int T(x',x)I(x')dx,
\]

where the kernel function \( T(x',x) \) is called the transfer function of the observation. Obviously, the goal is to determine \( I(x) \) from a measured \( S(x') \).

Equation (2) suggests the way to calculate the transfer function \( T(x',x) \), since the choice of the emissivity \( I(x) = \delta(x-x'') \) gives \( S(x')=T(x',x'') \), where \( x'' \) scans all possible values of \( x \). For a given measurement configuration, the \( T(x',x) \) transfer function of observation can indeed be calculated by simulating the observation of virtual light sources on the \( x'' \) flux surfaces for a whole range of \( x'' \) values, as it is illustrated in Fig. 1.

In Fig. 2, two transfer functions are contour plotted, illustrating the features of the deviation from an ideal measurement that would give a \( \delta(x'-x) \)-like transfer function fully centered upon the diagonal. Figure 2(a) corresponds to an unfavorable setup, when the lines of sight are quite far from tangential to the flux surfaces, in contrast to Fig. 2(b). A horizontal cut of the former transfer function is a wide Gaussian-type curve, showing that a detector segment in a given \( x' \) position collects the information from a broad range of spatial coordinate \( x \). The closer the lines of sight to tangential are to the flux surfaces at the beam position, the more local the measurement is.

The effect caused by the broadening of the transfer function, due to finite thickness of the beam, on the measured light profile and the corresponding density profile is illustrated for a quite unfavorable but still realistic case. The angle between the lines of sight and the flux surfaces, which we call observation angle, is approximately 45° on average at the beam position. The observation system is located 0.45 m far from the observed region. A Gaussian current distribution beam with full width at half maximum of 1.2 cm is injected into a high density plasma with pedestal. For this case the transfer function is similar to Fig. 2(a), and the corresponding light profiles are presented in Figs. 3(a) and 3(b). The emissivity distribution of an infinitesimally fine beam \( I(x) \) would give the best measurement of the electron density on the beam axis, up to the accuracy of the density profile reconstruction method. This strictly local measurement is referred as ideal (solid line). In reality we measure a light profile (measured, dotted line) affected by the finite beam width, which is smoother compared to the ideal. Before the density calculation, the measured profile is corrected to the spatially slowly varying geometrical efficiency factor giving the profile labeled as “calibrated” (dashed line). Note that the density reconstruction does not require the absolute value of the emissivity, only the shape of the light profile.

The relative differences from the ideal profile with respect to the maximum intensity are plotted in Fig. 3(b). Note that while the relative difference between the ideal and the

 FIG. 1. (Color online) Construction of the transfer function of the observation. \( x \) is the coordinate along the beam axis, which is one-to-one mapped to \( x' \) through the lines of sight crossing the axis. The image \( S(x') \) of a light source being on the flux surface poked by the axis at \( x'' \) gives \( T(x',x=x'') \).

 FIG. 2. Simulated transfer functions. The observation angle is (a) high or (b) small.
calibrated profiles is 5 at. % x = 5 cm, the maximum relative error of the density profile calculated from the calibrated light profile is 23% within the same range as it is shown on Figs. 3(c) and 3(d), where the corresponding density profiles are plotted together with the differences from the density profile used as input to the light profile calculations (original, dashed-dotted line). The density profiles are calculated by the ABSOLUT code.6 The corrected (long dashed line) curves show the result of the emission reconstruction correction method, which is introduced in the next section.

IV. EMISSION RECONSTRUCTION

A deconvolution-based method can be introduced for the correction of undesired smoothing effect due to the observation of a finite width beam addressed in the previous section. The method uses the properties of the beam evolution and the transfer function, and assumes that the plasma parameters are flux functions in the spatial scale of the beam width.

Two essential aspects determine the characteristics of the transfer function. On the one hand, the fact that we integrate over a range of flux surfaces, as the line of sight goes through the beam, gives the extradiagonal elements if we represent the discretized functions as a matrix. On the other hand, the geometrical efficiency of detection, the main factor of which is the variation in the solid angle of observation along the beam, is responsible for the slow trends in the magnitude. The latter effect would remain even if we used an ideal beam, and thus it is usually taken into consideration in the 1D calculations.

The two above effects are nearly independent of each other, thus the transfer function can be separated

\[
S(x') = \int T(x',x)I(x)dx \approx \int p(x)\tau(x-x)I(x)dx
\]

where, in the first step, we introduced a slowly varying function \(p(x)\) containing the geometrical efficiency factors, and the effect of the integration along the lines of sight is represented by \(\tau(x-x')\). Note that in an ideal measurement, \(\tau(x-x) = \delta(x-x')\). In the second step, \(\tau(x-x)\) is approximated by a convolution kernel \(\tau(x-x')\). This approximation means that the width of \(\tau(x-x)\) is independent of \(x\), being valid if the observation angle does not vary too much in the observed region. In the third step we used that \(p(x)\) is a slowly varying function of \(x\) compared to \(\tau(x-x')\).

We introduce the calibrated light profile \(S'(x') = \int S(x')/p(x')\), where the geometrical efficiency along the beam axis \(p(x')\) is calculated by the simulation, but otherwise an easily measurable quantity. The convolution kernel can be estimated by \(\tau(x-x') = T(x-x+x_c)/p(x_c)\), with \(x_c\) being in the middle of the range relevant from the density profile calculation point of view.

In order to invert a convolution, it is expedient to consider the problem in Fourier space. The convolution theorem gives

\[
\hat{S}'(k) = \hat{\tau}(k)\hat{I}(k),
\]

where the “hat” denotes the Fourier transform of a function. If \(\hat{\tau}(k) \neq 0\) for any \(k\) the problem would be solved, since then the inverse Fourier transform of

\[
\hat{I}(k) = \hat{S}'(k)/\hat{\tau}(k)
\]

would give the desired solution \(I(x)\). In reality, \(\tau(x-x)\) has a typical scale length, which is comparable to the beam width. Therefore, there is a finite \(k\), above which \(\hat{\tau}(k)\) drops rapidly, which we denote by \(k_r\). Division by such a \(\hat{\tau}(k)\) according to Eq. (5) would amplify the high wave number part of the noise present in the spectrum of the measured light profile. The spectrum of the measured light profile also decays exponentially due to the finite spontaneous decay time of the atomic states, but reaches the noise level at a frequency \(k \approx k_r\) intrinsically much lower than \(k_r\). To get around the problem we can zero out 1/\(\hat{\tau}(k)\) above \(k > k_r\) before the multiplication with \(\hat{S}'(k)\).

Typical wave number spectra of the emission reconstruction are plotted on Fig. 4. The corrected light profile spectrum plotted by the dashed line can be expressed as \(\Theta(k)\hat{S}'(k)/\hat{\tau}(k)\), where \(\Theta(k)\) is the Heaviside function. It decays rapidly with a slope determined by the spontaneous decay frequency of the observed transition, while the calibrated spectrum \(\hat{S}'\) (solid line) decays even somewhat faster due to the decaying \(\hat{\tau}\) transmission of the observation in wave number space (dotted line). The deconvolution function \(\Theta(k)\hat{S}'(k)/\hat{\tau}(k)\) is plotted with dashed-dotted line.

As we will see, there are measurement configurations and plasma parameters profiles when the smoothing effect of
the observation does not cause significant errors in the reconstructed density profile. However the quantification of the transmission of the system is still of importance, if the radial wave number spectrum of a BES fluctuation measurement is to be investigated. The smallest measurable spatial scale is limited by the effect of the finite lifetime of the atomic state, although there is a considerable part of the wave number spectrum that can be underestimated by not considering the transmission.

V. REALISTIC NUMERICAL TESTS

In the present section, the effects of the finite beam width on the light profile and the corresponding density profile are investigated in simulations for the COMPASS and the TEXTOR tokamaks. The undesired smoothing of the light profile due to the integration along the lines of sight through different stages of beam evolution is corrected by the emission reconstruction method introduced in the previous section.

The COMPASS simulations are based on the plasma parameter profiles of the $1.2 \times 10^{20}$ m$^{-3}$ central electron density H-mode shot 30866 (156 ms). The 40 keV Li/Na beam is injected in the outboard midplane, while the observation system is located in the same poloidal position at the high field side or middle top ports. The latter setup, which is the planned one for the COMPASS reinstalled at Prague recently, is shown in an output file of RENATE, Fig. 5, where the emissivity of the Li beam is contour plotted. Additionally, the magnetic geometry and the lines of sight together with the vacuum chamber are also indicated.

A lower density ($3.2 \times 10^{19}$ m$^{-3}$) shot 107242 (2730 ms) is taken from the TEXTOR circular tokamak, where the 35 keV Li beam is observed from a low field side (LFS) top port through a periscope system with quite high observation angle. The TEXTOR Li-BES setup is shown in Fig. 6. The corresponding sodium calculations are based on similar measurement geometry and the assumption of a 40 keV Na beam.

There are two aspects of the measurement playing an important role in the enhancement in the finite beam width effects: the observation angle and the scale length of the light profile compared to the beam width. The first one is mainly determined by the measurement geometry apart from the case of significantly variable magnetic geometry devices. In certain cases, the location of the observation system cannot be chosen to give optimal observation angle because of technical constraints, such as on the TEXTOR setup. The second aspect depends on the beam width, which is determined by the ion optics, and also on the plasma parameter profiles and the beam material determining the beam evolution.

The finite beam width effects on the light profile were discussed in the end of Sec. III and were illustrated on Fig. 3 showing a simulated Li-BES measurement on the COMPASS tokamak with LFS observation. In this case, besides the unfavorable observation direction, the relatively steep density gradient also enhances the smoothing of the light profile, since the characteristic length scale of the light profile—the distance between the point where the beam enters the plasma and the light profile maximum—is only 5 cm, comparable to the beam width. The varying geometrical ef-
ficiency has a smaller effect according to the measured and calibrated curves on Fig. 3(b). This difference is even more pronounced in the reconstructed density profiles calculated by the ABSOLUT code, as seen on Figs. 3(c) and 3(d). Here, the differences are measured with respect to the original density profile that was the input for the light profile calculations, which enables to show the accuracy of the density calculation method, as even the ideal density profile has a certain error. The calibrated density profile is also smoothed compared to the ideal, significantly underestimating the density at the pedestal region. However, its relative systematic error is approximately four times larger than that of the corresponding light profile.

The result of the emission reconstruction calculation from the measured light profile and the corresponding density profile is plotted with a long dashed line (corrected). The relative error of the corrected profile is only 1% at the maximum in contrast to the 10% maximum error in the calibrated one, or 5% in the region of interest from density profile calculation point of view. This error in the corrected profile causes 6% error in the density profile, which is the same magnitude as the error due to the imperfections of the density calculation.

According to the current plans, the observation system of the COMPASS BES will be installed into the middle top port, as in Fig. 5, which is more favorable, since the angle between the flux surfaces and the lines of sight is only ~25° on average. However the systematic error for the same parameters is still not negligible, more than 15% (see Fig. 7). The error in the calibrated light profile follows the same pattern as in the previous case, however the overestimation outside the 4–5 cm range is considerably lower. The calibrated light profile has more than four times higher error than the corrected one, and the improvement for the density profiles is a factor of 3, although as the ideal density profile shows, the improvement is limited again by the accuracy of the density calculation.

As we pointed out, not only the observation direction but the scale length of the light profile is also important in the finite beam effects of a density profile measurement. The scale length is affected on the one hand by the plasma parameter distributions along the beam line, mainly the electron density distribution and slightly by the beam energy, the temperature profile and impurity concentrations, and on the other hand the beam material. In the TEXTOR test case the density gradients are much lower than for the COMPASS case, giving three times longer light profile scale length, much higher than the beam width (see Fig. 6). Although the observation angle is 50° the finite beam width effects are negligible.

However, the emission reconstruction method can potentially be used for the density profile measurement during a two-dimensional fluctuation measurement. The beam scans in the poloidal direction (at 400 kHz on TEXTOR) in order to give good poloidal velocity resolution of the turbulent structures, while the concurrent profile measurement has a much lower sampling rate, thus the different beam positions are seen simultaneously. The configuration is equivalent with the observation of only one quite thick beam (~5 cm), which clearly gives non-negligible finite beam width effects.

The effect of the beam material is illustrated by Na beam simulations in Figs. 8 and 9. For a smaller ionization energy atom, such as Na, the light profile becomes shorter for the same plasma parameter distributions, which enhances the finite beam width effects. For middle top port observation COMPASS case using Na beam, shown in Fig. 8, the overestimation of the light profile is more than two times higher than with Li beam (Fig. 7), while the error of the corrected
light profile is the same in both cases. In the TEXTOR case, shown in Fig. 9, a similar trend can be seen. For Li beam the systematic error of the measured profile was clearly dominated by the geometrical efficiency effects, but for Na the contribution of the finite beam width effects gives the half of this error.

VI. ERROR ESTIMATION FOR LI-BES MEASUREMENTS

After having investigated the characteristics of the error caused by the finite beam width on typical simulated measurements of the COMPASS and TEXTOR tokamaks, it is instructive to quantitatively measure the different factors affecting the error, on the basis of which it can be estimated for different measurements and configurations. In this chapter we focus only on measurements using lithium as beam material, which is the most common for diagnostic purpose beams.

Assuming that the evolution of plasma parameters follow the flux surfaces, for given plasma parameters, we expect the error to be proportional to the typical width of the range of flux surfaces we integrate through along the line of sight. This width can be estimated as \( b|\sin(\alpha)/\cos(\alpha-\beta)| \), where \( b \) is the full width at half maximum of the beam, \( \alpha \) is the observation angle, and \( \beta = \pi/2 - \beta' \) if \( \beta' \) is the angle between the beam axis and the flux surfaces (see Fig. 10). The trigonometric expression gives no error when the lines of sight are tangential to the flux surfaces, and diverges as they become parallel to the beam axis, although the latter configuration is quite unnatural. Here, we note that the first assumption of the paragraph breaks down if \( \beta' \) is too small. If the axis is perpendicular to the flux surfaces on average, the expression reduces to \( b|\tan \alpha| \). These angles are average values over a range along the beam relevant to the measurement.

Experience with several simulations for a wide range of plasma parameter profiles showed that the most important parameter affecting the magnitude of the systematic error is the electron density at the maximum of the light profile \( n_{e \max} \), and that the dependence of the maximum relative error in electron density on this parameter is approximately linear. The maximum relative error usually occurs in the vicinity of the light profile maximum, and due to the fact that the light profile smoothing effect of the observation mainly lowers the logarithmic derivative of the light profile, the error always appears as an underestimation of the electron density.

Investigating the effect of the electron temperature on the error, we found that the relative variation in the error for different temperature profiles is comparable to the relative variation in the quantity \( f_{n_e}(x)\frac{\partial}{\partial x}\int n_e \,dx \), where we integrate from the point where the beam enters the plasma to the maximum of the light profile. This variation is negligibly small for an experimentally relevant range of SOL/edge temperatures, and therefore can be neglected. Finally the error shows a relatively weak linear dependence on \( Z_{\text{eff}} \). Thus, the maximum relative error can be estimated as

\[
\text{max}(\frac{\Delta n_e}{n_e}) = C n_{e \max} f_{Z_{\text{eff}}} |\sin(\alpha)/\cos(\alpha-\beta)|, \tag{6}
\]

where \( f_{Z_{\text{eff}}} = 0.068 \) \( Z_{\text{eff}} + 0.9 \), \( n_{e \max} \) is given in \( 10^{19} \) \( \text{m}^{-3} \), \( b \) is in \( m \), and the constant \( C \) is found by linear fitting to be 3.8.

In Fig. 11, maximum relative errors are plotted against \( n_{e \max} f_{Z_{\text{eff}}} |\tan(\alpha)| \) for various plasma parameter profiles, observation angles, and beam widths. In these cases the beam axis is chosen to be perpendicular to the flux surfaces. The errors are calculated for the densities corresponding to the calibrated light profiles, and accordingly contain only the effects due to the finite beam width. The considerable deviation of the errors with respect to the estimation indicated by the line is partly due to the finite accuracy of the density calculation method, but profile effects of the plasma parameter distributions, which would be difficult to parameterize, also play an important role.

Equation (6) predicts 21% relative error for the COMPASS case shown in Fig. 3, where we found 23% and 4.5% for the simulated TEXTOR measurement in Fig. 6, which is indeed lower than the achieved accuracy of the density calculation for that case. The formula can be used to give a rough estimate to the error expected for a measurement configuration. However, it is not capable of giving accurate implications to the error of a certain measurement, and in particular, cannot be used for correction of the error. For that, one has to resort to the comprehensive simulation of the BES measurement (e.g., using the RENATE code).
VII. CONCLUSIONS

In the present paper a deconvolution-based correction method of alkali BES density profile measurements has been presented and demonstrated in simulated measurements on the TEXTOR and the COMPASS tokamak using the actual/planned BES configuration, respectively. We found that in setups where the line of sight is far from tangential to the flux surfaces at the beam position, the observation can cause an undesired smoothing of the light profile, which results in an underestimation of the reconstructed density profile, up to 15%–20% for realistic cases. The systematic error caused by the finite beam width is larger for higher electron densities and for sodium or beam materials with even lower binding energies.

The systematic error investigated here caused by the integration along the lines of sight is important, since it causes information losses on the fine structure of the profile, and leads to the underestimation of the pedestal density gradient, degrading the capabilities of the BES measurement in very important fields such as investigation of L–H transitions. A general estimation of the maximal relative error in electron density is presented, reflecting that the error is proportional to the electron density at the light profile maximum and shows a linear dependence on \( Z_{\text{eff}} \) while its temperature dependence is negligible for experimentally relevant temperatures. Furthermore, it is proportional to the beam width and to the tangent of the average angle between the flux surfaces and the lines of sight. The maximum relative error regularly occurs near the light profile maximum and always appears as an underestimation of the electron density.

The transfer function of the observation, playing crucial role in the emission reconstruction, is calculated by the RENATE alkali BES simulation code. It takes the finite beam width and all basic properties of the measurement into account, assuming that the plasma parameters are flux functions on the scale of the beam width. Separating the transfer function into a slowly varying part due to geometrical efficiency effects and a convolution kernel describing the smoothing of the light profile, the problem can be reduced to a simple algebraic equation in wave number space.

The deconvolution method gives a good estimate of the emissivity on the beam axis from the measured light profile, so that the level of the remaining error due to the observation is in the order of the accuracy of the density profile reconstruction algorithm (in our case the ABSOLUT code\(^6\)). The method allows the use of the 1D density calculation methods even for the configurations where the finite width of the beam is not negligible.

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