Electron kinetics in a cooling plasma

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The distribution function of suprathermal electrons in a slowly cooling plasma is calculated by an asymptotic expansion in the cooling rate divided by the collision frequency. Since the collision frequency decreases with increasing velocity, a high-energy tail forms in the electron distribution function as the bulk population cools down. Under certain simplifying assumptions (slow cooling, constant density, Born approximation of cross sections), the distribution function evolves to a self-similar state where the tail is inversely proportional to the cube of the velocity. Its practical consequences are discussed briefly. © *2004 American Institute of Physics*.
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I. INTRODUCTION

Some of the most important and interesting phenomena in plasma physics involve a sudden or gradual cooling of the plasma electrons. For instance, in a solar flare the plasma in a coronal loop is first heated (by means that are hotly debated) and then cools down by heat conduction to the photosphere and by emission of radiation. The latter mechanism is dominant if the temperature is low enough, and results in a temperature evolution given approximately by $dT/dt \propto -T^{\alpha}$ with $\alpha \simeq -1/2$ in the most important temperature range,[1] so that

$$T(t) = T_0(1 - t/t_0)^{2/3}. \tag{1}$$

In laboratory plasma physics, tokamak discharges are frequently terminated by a disruptive instability, whereupon the plasma interacts with the wall and impurity ions enter the plasma, causing a catastrophic loss of energy due to the excitation and ionization of these ions.[2] (It is likely that heat conduction along open field lines connecting the plasma to the wall can also play an important role in cooling the plasma, but we do not consider this situation.) In an idealized picture of this so-called "thermal quench," the temperature evolution can be expected to follow Eq. (1) approximately. This is because the differential cross sections for excitation and ionization by electron impact are inversely proportional to the electron energy in the Born approximation where this energy far exceeds the relevant thresholds.[3] As long as the temperature is high enough and the density remains approximately constant, the energy loss rate from the plasma is thus proportional to $T^{-1/2}$ and the temperature follows Eq. (1). This scaling also agrees with the general expression for the energy loss rate of fast charged particles moving through matter.[4,5] Other examples of laboratory plasmas with cooling electrons include laser plasmas after the laser has been switched off,[6] and tokamak or stellarator plasma that has been expelled from a hotter into a colder region, such as the scrape-off layer after an edge-localized mode.[7]

In this paper, we calculate the distribution function of high-energy electrons in plasmas such as these, where the energy loss channel mostly involves thermal electrons, so that suprathermal particles lose energy primarily through collisions with the thermal population. As is appropriate for the applications just mentioned, the cooling is assumed to occur on a time scale longer than the collision time, so that the bulk of the electron distribution remains approximately Maxwellian. However, since the collision frequency decreases with increasing energy, fast electrons lose energy less quickly than slower ones and a high-energy tail therefore develops as the plasma cools down, even if the plasma were Maxwellian before it started to cool down. It is found that, after an initial transient, the structure of the high-energy tail becomes independent of initial conditions and evolves to a universal, self-similar shape if the bulk temperature evolves according to Eq. (1). At high energies, the distribution function follows a power law, $f \propto v^{-3}$. The presence of the high-energy tail can affect important properties of the plasma. In the solar context, for example, the part of the emission spectrum caused by the tail is sensitive to its structure, which has obvious observational consequences. The electrical and thermal conductivities of the plasma along the magnetic field can also increase substantially since the current and heat flux are carried by particles of relatively high energy. In tokamak disruptions, the presence of the tail can lead to a highly enhanced production of "runaway" electrons. The primary runaway electron generation rate is proportional to the number of electrons in the tail of the distribution function. It is therefore normally exponentially small in the electric field,[8,9] but would be very much enhanced in a cooling plasma. Besides spontaneous disruptions, this could also be important in fast plasma shutdown events caused by gas or pellet injection.[10–13]

II. EXPANSION OF THE KINETIC EQUATION

The calculation proceeds from the pitch-angle averaged kinetic equation for fast electrons in a homogeneous plasma,[14]

$$\frac{\partial f}{\partial t} = \frac{\hat{\nu}_{ee}\nu_T^3}{\nu^2}\frac{\partial}{\partial \nu}\left(f + \frac{T}{m\nu}\frac{\partial f}{\partial \nu}\right) + \frac{1}{\nu^2}\frac{\partial}{\partial \nu}(\nu_{ez}\nu^3 f), \quad (2)$$

where $\nu_T(t)=[2T(t)/m]^{1/2}$ is the thermal speed corresponding to Eq. (1) and $\hat{\nu}_{ee}=ne^4 \ln \Lambda/4\pi\epsilon_0^2 m\nu_T^3$ the collision frequency at that speed. The last term describes energy loss due to inelastic collisions involving ionization and excitation of impurities. For simplicity, we have written this term in the Fokker–Planck approximation where the energy lost in each ionization or excitation event is small relative to the energy of the impacting electron. The time and velocity dependence of the frequency $\nu_{ez}(\nu,t)$ depends on the impurity content and distribution over charge states, but always decays at high energies as $\nu_{ez}(\nu)\propto \nu^{-3}$ in the Born approximation mentioned above. As we shall see, this implies that this term only has a small effect on the high-energy tail of the electron distribution function.

It is convenient to normalize time to $\hat{\nu}_{ee}^{-1}$ by writing $ds=\hat{\nu}_{ee}dt$, the velocity to ν_T by writing $x=\nu/\nu_T$, and the distribution function by writing $\tilde{f}=f\nu_T^3\pi^{3/2}/n_e$, so that the kinetic equation (2) becomes

$$\frac{\partial \tilde{f}}{\partial s} + 3\delta\tilde{f} + \delta x\frac{\partial \tilde{f}}{\partial x} = \frac{1}{x^2}\frac{\partial}{\partial x}\left([1+\delta r(x,t)]\tilde{f} + \frac{1}{2x}\frac{\partial \tilde{f}}{\partial x}\right) \quad (3)$$

with

$$r(x,t) = \frac{x^3\nu_{ez}}{\delta\hat{\nu}_{ee}}$$

and

$$\delta = -\frac{1}{2\hat{\nu}_{ee}}\frac{d\ln T}{dt}.$$

When the temperature evolves according to Eq. (1), δ is independent of time and can be written in terms of the initial collision frequency and the cooling time as $\delta=1/[3\hat{\nu}_{ee}(0)t_0]$. The temperature then evolves as $T(t)=T(0)e^{-2\delta s}$.

As indicated above, we solve Eq. (3) in the limit of small δ, so that the plasma is assumed to cool down slowly relative to the collision time. For simplicity, we also take the initial condition to be Maxwellian, $\tilde{f}(x,0)=e^{-x^2}$, but as we shall see the final state is independent of this assumption. The perturbation analysis proceeds along the lines of a famous calculation by Kruskal and Bernstein of runaway electron generation,[9] which involves matching asymptotic expansions in five separate regions of velocity space. The number of regions and their character is similar although their location in velocity space is somewhat different, but, unlike that calculation, the present problem is soluble to lowest order in all five regions (at least for the final state, see below). A complete analytical solution for the asymptotic limit $\delta \ll 1$ is thus available.

Region I: $x\sim 1$. A straightforward regular perturbation expansion,

$$\tilde{f} = f_0 + \delta f_1 + \cdots, \quad (4)$$

gives $f_0=e^{-x^2}$ and

$$f_1 = \left(\frac{2x^5}{5} - 2\int_0^x r(x',\infty)x'dx'\right)e^{-x^2}$$

for $s\to\infty$. As expected, the distribution is almost Maxwellian and the cooling has the effect of enhancing the relative number of high-energy particles. However, the term δf_1 is smaller than f_0 only for $x\ll\delta^{-1/5}$, and the expansion breaks down for higher energies. A different procedure is thus required for calculating the distribution function at these energies.

Region II: $x\sim\delta^{-1/5}$. To treat this region we introduce an appropriately stretched velocity variable $u=x\delta^{1/5}$ and note that the Maxwellian

$$f_0 = e^{-u^2/\delta^{2/5}}$$

is not expandable in a power series of δ. An expansion of the type (4) is therefore of no use, so instead we consider the equation for $F=\ln\tilde{f}$, which is nonlinear,

$$\frac{\partial F}{\partial s} + \delta\left(3 - \frac{\partial r/\partial x}{x^2}\right) + \delta\left(x - \frac{r}{x^2}\right)\frac{\partial F}{\partial x} = \frac{1}{x^2}\left(1-\frac{1}{2x^2}\right)\frac{\partial F}{\partial x} + \frac{1}{2x^3}\left[\left(\frac{\partial F}{\partial x}\right)^2 + \frac{\partial^2 F}{\partial x^2}\right], \quad (5)$$

and expand

$$F = \frac{1}{\delta^{2/5}}(F^0 + \delta^{2/5}F^1 + \cdots). \quad (6)$$

In lowest order we then obtain

$$\frac{\partial F^0}{\partial s'} = \frac{1}{u^2}\frac{\partial F^0}{\partial u} + \frac{1}{2u^3}\left(\frac{\partial F^0}{\partial u}\right)^2, \quad (7)$$

where we have rescaled the time variable by writing $s'=\delta^{3/5}s$. This choice of "maximal" ordering is dictated by the need to make the s derivative appear in the same order as the leading u derivatives. The solution of Eq. (7) is the Maxwellian, $F^0(u,s')=-u^2$, as expected, and in next order an equation for the tail is obtained,

$$\frac{\partial F^1}{\partial s'} + \frac{1}{u^2}\frac{\partial F^1}{\partial u} = 2u^2.$$

When the initial distribution function is Maxwellian, so that $F^1(u,0)=0$, the solution is

$$F^1(u,s') = \frac{2u^5}{5} - \frac{2}{5}(u^3-3s')^{5/3}\theta(u^3-3s'),$$

where θ is the Heaviside step function. This indicates that the final state $F^1(u,\infty)=2u^5/5$ is reached in a finite time s'_*. The time this takes increases with velocity,

$$s'_* = u^3/3,$$

because of the velocity dependence of the collision operator. Note that, in contrast to Region I, the number of high-energy particles is enhanced significantly in Region II because $\tilde{f}/e^{-x^2} \simeq e^{F^1}$ can be large although $F^1 \ll F^0$.

The expansion (6) relies on $\delta^{2/5}F^1$ being smaller than F^0, which is true only for $x \ll \delta^{-1/3}$. Again, the validity range of the expansion is limited and the solution has a different structure at higher energies.

Region III: $x \leqslant \delta^{-1/3}$. To analyze the region $x \sim \delta^{-1/3}$ we again rescale the velocity, this time by writing $y = x\delta^{1/3}$. Equation (5) then becomes

$$\frac{\partial F}{\partial \tau} + y\frac{\partial F}{\partial y} + 3 - \frac{\delta}{y^2}\left(r\frac{\partial F}{\partial y} + \frac{\partial r}{\partial y}\right) = \frac{1}{y^2}\left(1 - \frac{\delta^{2/3}}{2y^2}\right)\frac{\partial F}{\partial y} + \frac{\delta^{2/3}}{2y^3}\left[\left(\frac{\partial F}{\partial y}\right)^2 + \frac{\partial^2 F}{\partial y^2}\right], \tag{8}$$

where we have written $\tau = s\delta$, so that the leading terms with derivatives with respect to time and velocity again appear in the same order. Expanding the solution in powers of $\delta^{2/3}$,

$$F = \frac{1}{\delta^{2/3}}(F_0 + \delta^{2/3}F_1 + \cdots), \tag{9}$$

gives in lowest order

$$\frac{\partial F_0}{\partial \tau} = \left(\frac{1}{y^2} - y\right)\frac{\partial F_0}{\partial y} + \frac{1}{2y^3}\left(\frac{\partial F_0}{\partial y}\right)^2. \tag{10}$$

Thus, unlike the Kruskal–Bernstein runaway calculation,[9] we obtain a nonlinear equation in this region. Analytically, we can only determine the full time dependence implicitly, but the final state is easily calculated by letting $\partial F_0/\partial \tau = 0$, which gives the two solutions

$$F_0(y,\infty) = -y^2 + \frac{2y^5}{5} \tag{11}$$

and

$$F_0(y,\infty) = \text{const}. \tag{12}$$

It would appear that only the first solution is relevant since only this solution matches to the lower-energy regions. However, the first solution is in fact approached only for $y < 1$, while the second one holds for $y > 1$. To see this, we write

$$F_0(y,\tau) = -\frac{y^2}{2} + \frac{y^5}{5} + g,$$

so that

$$\frac{\partial g}{\partial \tau} = \frac{1}{2}\left(\frac{\partial g}{\partial z}\right)^2 + p(z)$$

with $z = 2y^{5/2}/5$ and

$$p(z) = -\frac{1}{2}\left[\left(\frac{2}{5z}\right)^{1/5} - \frac{5z}{2}\right]^2.$$

Hence it follows that $h(z,\tau) = -\partial g/\partial z$ satisfies

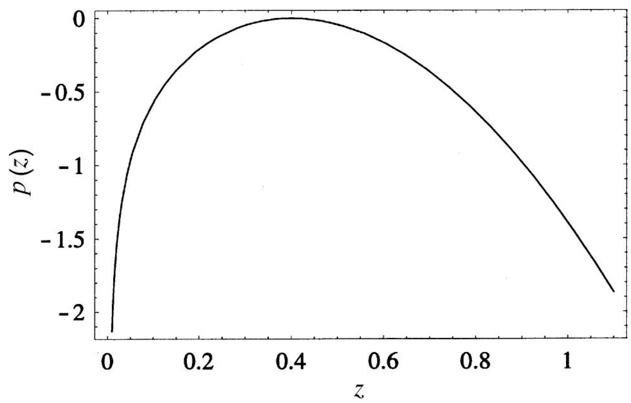

FIG. 1. Shape of the potential $p(z)$. The time it takes for the final state to be reached at the position z is equal to the time required for a particle to travel from 0 to z if its total energy $E = \dot{z}^2/2 + p(z)$ vanishes. This time is infinite for $z > 2/5$ since the particle then comes to rest at the top of the hill, $z = 2/5$.

$$\frac{\partial h}{\partial \tau} + h\frac{\partial h}{\partial z} = -p'(z), \tag{13}$$

which is the equation of motion for a one-dimensional inviscid fluid with pressure $p(z)$. This is mathematically equivalent to particle motion in the potential $p(z)$, see Fig. 1. The fluid velocity $h(z,\tau)$ is equal to the velocity of a particle moving in this potential and arriving at the position z at the time τ, if its initial velocity was $h(z_0,0)$ and z_0 was the initial position. In other words, the solution of Eq. (13) can be found by constructing the particle trajectory determined by

$$\ddot{Z}(\tau) = -p'(Z),$$
$$Z(0) = z_0, \tag{14}$$
$$\dot{Z}(0) = h(z_0,0),$$

where the overhead dots denote derivatives with respect to τ. For given z and τ, if z_0 is chosen so that $Z(\tau) = z$ then the solution of Eq. (13) is $h(z,\tau) = \dot{Z}(\tau)$. Because the energy

$$E(z_0) = \frac{\dot{Z}^2}{2} + p(Z) = \frac{h^2(z_0,0)}{2} + p(z_0)$$

is conserved during the motion, the solution of Eq. (14) is given implicity by

$$\tau(Z,z_0) = \int_{z_0}^{Z} \frac{dz}{\sqrt{2[E(z_0) - p(z)]}}, \tag{15}$$

where $E = 2(5z_0/2)^{4/5}$ since

$$h(z,0) = -\frac{\partial g(z,0)}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y^2}{2} + \frac{y^5}{5}\right) = \left(\frac{2}{5z}\right)^{1/5} + \frac{5z}{2}$$

if the initial distribution function is Maxwellian, $F_0(y,0) = -y^2$.

These relations determine implicitly the solution of Eq. (10), and imply that the final state Eq. (11) is reached in the time τ_* given by $\tau(z,0)$, i.e.,

$$\tau_*(z) = \int_0^z \frac{dz'}{(2/5z')^{1/5} - 5z'/2}. \quad (16)$$

This is because a particle released in the potential $p(z)$ at $z_0=0$ has zero energy $E=0$ and will therefore arrive at z at the time τ_* with the velocity $\sqrt{-2p(z)}$. This implies that

$$h(z,\tau_*(z)) = \left| \left(\frac{2}{5z}\right)^{1/5} - \frac{5z}{2} \right|, \quad (17)$$

so that

$$F_0(y,\tau_*) = -y^2 + \frac{2y^5}{5}$$

as in Eq. (11). The time (16) is finite if $z<2/5$, so that the final state Eq. (11) is reached in finite time if $y<1$, just as in Region II. In contrast, for $y>1$ it takes infinite time to reach the final state since in Eq. (15),

$$\lim_{z_0\to 0} \tau(z,z_0) = \infty$$

if $z>2/5$, see Fig. 1.

The region $y>1$ is also different from $y<1$ in another way. Because of the absolute value taken in Eq. (17), the final state which is reached as $\tau\to\infty$ in Eq. (15) is not Eq. (11) but Eq. (12). In lowest order, the distribution function approaches a constant if $y>1$. In next order, one finds from Eq. (8)

$$F_1(y,\infty) = \begin{cases} 0, & y<1 \\ -\ln(y^3-1), & y>1. \end{cases} \quad (18)$$

The singularity at $y=1$ indicates that the perturbation expansion (9) breaks down in a narrow region around $y=1$. The expansion is initially valid everywhere, but a boundary layer at $y=1$ develops as $\tau\to\infty$.

Region IV: $x \simeq \delta^{-1/3}$. The width of the boundary layer is set by the requirement $\partial F_0/\partial y > \delta^{2/3}\partial F_1/\partial y$ in the expansion (9), which is violated for $y-1\sim\delta^{1/3}$. Thus writing $y=1+\delta^{1/3}\omega$ gives in lowest order

$$\frac{\partial\tilde f}{\partial s} + 3\left(\tilde f + \omega\frac{\partial \tilde f}{\partial\omega}\right) = \frac{1}{2}\frac{\partial^2\tilde f}{\partial\omega^2},$$

and it follows that the final state, which is calculated by setting $\partial\tilde f/\partial s=0$, is

$$\tilde f(\omega,\infty) = [c_1 - c_2\,\mathrm{erf}(\omega\sqrt{3})]e^{3\omega^2},$$

where c_1 and c_2 are integration constants. Requiring the solution to vanish for $\omega\to+\infty$ and to match the $y\to1^-$ limit of Eq. (11) for $\omega\to-\infty$ gives

$$c_1 = c_2 = \frac{1}{2}\exp\left(-\frac{3}{5\delta^{2/3}}\right),$$

so that the solution in this region becomes

$$\tilde f(\omega,\infty) = \tfrac{1}{2}e^{3(\omega^2-\delta^{-2/3}/5)}\,\mathrm{erfc}(\omega\sqrt{3}).$$

Region V: $x>\delta^{-1/3}$. On this side of the boundary layer, the solution given by Eqs. (12) and (18) holds. These equations show that the solution varies slowly with velocity, so that a regular perturbation expansion holds. Changing the independent variables in Eq. (3) to y and τ gives in lowest order

$$\frac{\partial\tilde f}{\partial\tau} + 3\tilde f + y\frac{\partial\tilde f}{\partial y} = \frac{1}{y^2}\frac{\partial\tilde f}{\partial y}.$$

This equation cannot be used to calculate the time evolution from the initial, Maxwellian state, which is not expandable in δ, but gives the final state as

$$\tilde f = \frac{c_3}{y^3-1} \quad (19)$$

in agreement with Eq. (18). The integration constant is chosen to match the $\omega\to+\infty$ limit of Region IV,

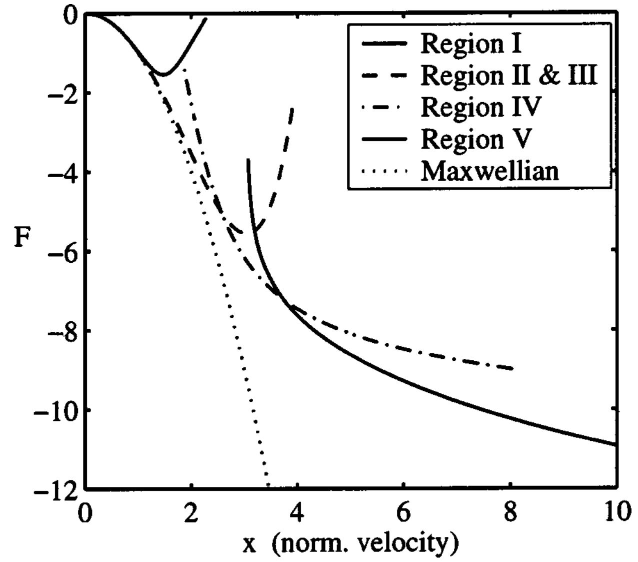

FIG. 2. Asymptotic distribution function in the various regions for $\delta=0.035$.

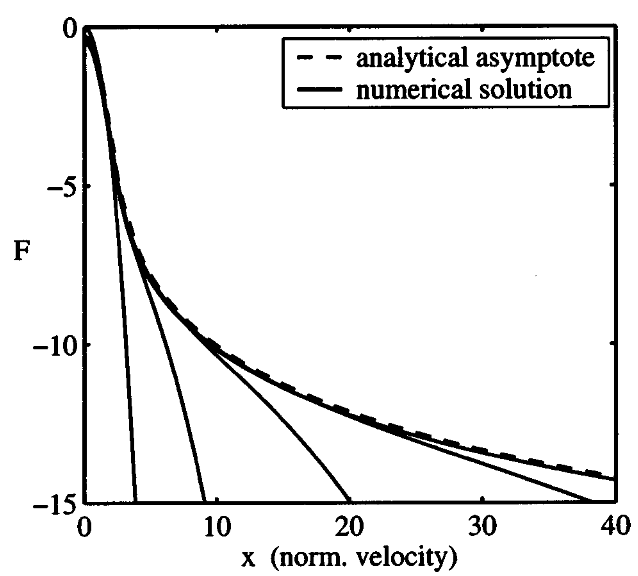

FIG. 3. Comparison of a direct numerical solution of Eq. (3) and the asymptotic end state given by Eq. (20) for $\delta=0.05$. The numerical solutions are, from left to right, given at the normalized times $s=0$, 20, 40, 60, and 80.

$$c_3 = \frac{\delta^{1/3}}{2}\left(\frac{3}{\pi}\right)^{1/2} \exp\left(-\frac{3}{5\delta^{2/3}}\right).$$

Equation (19) represents the asymptotic state corresponding to $\tau \to \infty$ but does not extend to arbitrarily high energies at fixed τ. It cannot, of course, since this distribution contains infinitely many particles at high energies. An estimate of the upper velocity limit of Eq. (19) can be obtained by evaluating Eq. (15) for small z_0 and large Z,

$$\tau(Z,z_0) \simeq \tfrac{2}{5} \ln Z - \tfrac{4}{15} \ln z_0 .$$

In Fig. 1, this is the time it takes for a particle to travel from $z_0 \ll 2/5$ to $Z \gg 2/5$. It follows that for $\tau \gg (2/5)\ln z \simeq \ln y$ the distribution function is given by Eqs. (12) and (18), i.e., by Eq. (19), while in the opposite limit, $\tau \ll \ln y$, the distribution function is still in its initial state. Indeed, the velocity $y = e^{\tau}$ corresponds to $v = (2T_0/m)^{1/2}\delta^{-1/3}$, which is the speed at which the collision time equals the cooling time (within a factor of order unity). The collision frequency in the region $y \gg e^{\tau}$ is thus so small that the particles have still not had time to react to the cooling of the bulk plasma.

Figure 2 shows how the different asymptotic expressions derived in this section for the final state, $s \to \infty$, match up to produce an overall smooth distribution function. In this figure, the slopes to do not match exactly because of the finite value chosen for δ, but the match becomes entirely smooth in the limit $\delta \to 0$.

III. CONCLUSIONS

The perturbation analysis in the preceding section shows that the distribution function of electrons in a plasma cooling down according to Eq. (1) is given approximately by

$$f(y,t) = n\left(\frac{m}{2\pi T}\right)^{3/2} \begin{cases} \exp\left[-\frac{1}{\delta^{2/3}}\left(y^2 - \frac{2y^5}{5}\right)\right], & y < 1 \\ \frac{1}{2}\exp\left[\frac{3}{\delta^{2/3}}\left((y-1)^2 - \frac{1}{5}\right)\right]\operatorname{erfc}\left(\frac{\sqrt{3}(y-1)}{\delta^{1/3}}\right), & y - 1 \sim \delta^{1/3} \\ \frac{\delta^{1/3}}{2}\left(\frac{3}{\pi}\right)^{1/2}\exp\left(-\frac{3}{5\delta^{2/3}}\right)\frac{1}{y^3-1}, & y > 1 \end{cases} \qquad (20)$$

for large t. Here the first three regions are represented by a single expression in lowest order. This final (but still self-similarly evolving) state does not depend on the initial distribution function, but in the case that the plasma is initially Maxwellian, the final state is reached in the finite time given by Eq. (16) for $y < 1$. Equation (20) agrees very well with direct numerical solutions of Eq. (3). An example is shown in Fig. 3. The agreement is remarkable given that the expansion parameter, which is $\delta^{2/5} = 0.3$ in Region II and $\delta^{2/3} = 0.14$ in Region III, is not very small. The numerical solution was found by straightforward discretization of Eq. (3) and was checked by comparing with a similar simulation using the ARENA Monte Carlo code.[15]

The distribution function given by Eq. (20) is self-similar in the sense that although its height increases and its width shrinks as the plasma cools down, its shape does not change with time when the distribution function is expressed in rescaled velocity variables, as we have done. The possibility of self-similar cooling was noticed in Ref. 16, where the solution of a slightly more general kinetic equation was found by numerical means and it was pointed out that the non-Maxwellian nature of the distribution function can have a dramatic effect on parallel transport. As remarked in the Introduction, it can also affect the radiation emitted by a cooling plasma, which could be of importance for the interpretation of observational data from solar flares.

Another important effect of the enhanced population of high-energy electrons may be on the generation of runaway electrons during tokamak disruptions. The number of runaways generated in these events is largely determined by the high-energy tail of the electron distribution function, which is likely to be enhanced for plasmas typical of present and future experiments, even under very moderate assumptions. This effect has been the subject of numerical simulations in connection with "killer pellets" experiments,[10,11] where a burst of runway production occurs as the plasma cools down. It may be especially important for disruptions in the International Thermonuclear Experimental Reactor,[17] where the projected electron temperature is around 20 keV and the collision time is therefore not very much shorter than the expected duration of the thermal quench in a disruption. Of course, the electron temperature does not necessarily follow Eq. (1) during a tokamak disruption where the impurity content changes rapidly with time, but the qualitative behavior of tail formation should be similar, as indeed found in Refs. 10 and 11. Quantitatively reliable results can only be obtained from a numerical solution of the kinetic equation, but the analytical solution presented here provides a useful approximate estimate as well as a rigorous benchmark on the numerics. This and other issues specifically connected with runaway generation will be discussed in a forthcoming paper.

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[1]S. K. Antiochus, Astrophys. J. **241**, 385 (1980); P. J. Cargill, J. T. Mariska, and S. K. Antiochos, *ibid.* **439**, 1034 (1995).
[2]J. A. Wesson, R. D. Gill, H. Hugon *et al.*, Nucl. Fusion **29**, 641 (1989).
[3]L. D. Landau and E. M. Lifshitz, in *Quantum Mechanics*, Course of Theoretical Physics Vol. 3, 3rd ed. (Pergamon, Oxford, 1977).
[4]L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, in *Electrodynamics of Continuous Media*, Course of Theoretical Physics Vol. 8, 2nd ed. (Pergamon, Oxford, 1984).
[5]J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
[6]E. Fourkal, V. Yu. Bychenkov, W. Rozmus, R. Sydora, C. Kirkby, C. E. Capjack, S. H. Glenzer, and H. A. Baldis, Phys. Plasmas **8**, 550 (2001); O. V. Batishchev, V. Yu. Bychenkov, F. Detering, W. Rozmus, R. Sydora, C. E. Capjack, and V. N. Novikov, *ibid.* **9**, 2302 (2002).
[7]D. J. Sigmar, A. A. Batishcheva, O. V. Batishchev, S. I. Krasheninnikov, and P. J. Catto, Contrib. Plasma Phys. **36**, 230 (1996); O. V. Batishchev, A. A. Batishcheva, P. J. Catto, S. I. Krasheninnikov, B. LaBombard, B. Lipshulz, and D. J. Sigmar, J. Nucl. Mater. **241–243**, 374 (1997).
[8]A. V. Gurevich, Sov. Phys. JETP **12**, 904 (1961).
[9]M. D. Kruskal, I. B. Bernstein, Princeton Plasma Physics Laboratory Report No. MATT-Q-20, 1962 (unpublished).
[10]S. C. Chiu, M. N. Rosenbluth, R. W. Harvey, and V. S. Chan, Nucl. Fusion **38**, 1711 (1998).
[11]R. W. Harvey, V. S. Chan, S. C. Chiu, T. E. Evans, M. N. Rosenbluth, and D. G. Whyte, Phys. Plasmas **7**, 4590 (2000).
[12]K. H. Finken, G. Mank, A. Krämer-Flecken, and R. Jaspers, Nucl. Fusion **41**, 1651 (2001).
[13]D. G. Whyte, T. C. Jernigan, D. A. Humphreys *et al.*, Phys. Rev. Lett. **89**, 055001 (2002).
[14]P. Helander and D. J. Sigmar, *Collisional Transport in Magnetized Plasmas* (Cambridge University Press, Cambridge, 2002).
[15]L.-G. Eriksson and P. Helander, Comput. Phys. Commun. **154**, 175 (2003).
[16]A. A. Batishcheva, O. V. Batishchev, M. M. Shoucri, S. I. Krasheninnikov, P. J. Catto, I. P. Shkarofsky, and D. J. Sigmar, Phys. Plasmas **3**, 1634 (1996).
[17]ITER Physics Basis, Nucl. Fusion **39**, 2137 (1999).