Neutral diffusion and anomalous effects on collisional ion flow shear in tokamaks

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Ion plasma flow and flow shear just inside the last closed flux surface of a tokamak can be strongly altered by neutral atoms and anomalous effects. For a collisional edge, neutrals modify the standard Pfirsch–Schlüter expression for the parallel ion flow through the strong coupling provided by ion–neutral collisions. Even for rather small neutral to plasma density ratios, the large diffusivity of the neutrals in combination with the ion–neutral coupling can directly modify the ion distribution function as well as cause neutral diffusion modifications to the parallel momentum constraint that determines the parallel ion flow. Direct modification of the ion distribution function only dominates at order unity aspect ratios, and was unimportant in an earlier plateau evaluation of the effects of neutrals on ion flow. Like the earlier work, anomalous effects are retained to maintain a steady state and demonstrate that large anomalous transport can alter neoclassical collisional ion flows. © *1998 American Institute of Physics.* [S1070-664X(98)01511-0]

I. INTRODUCTION

It is widely believed that the transition from the L to H mode (low to high confinement) in tokamaks involves turbulence stabilization by a sheared plasma flow.[1] It is therefore important to understand how the plasma flow arises and how it may differ from the neoclassical prediction because of the influence of neutral atoms[2] and anomalous diffusion.

Recent work in the plateau regime[3] corrected earlier work[4,5] by finding that the diffusion of neutral atoms affects the ion flow by momentum exchange due to ion–neutral charge exchange. To achieve a steady state, the inward neutral diffusion was assumed to be balanced by an outward anomalous transport. In the present work we analyze the effects of neutral diffusion and anomalous processes on tokamak ion flow in the Pfirsch–Schlüter regime, assuming that the mean-free path of neutrals is short. Plateau calculations always rely on a large aspect ratio expansion and are only valid in a restricted interval of collisionality. In practice, these assumptions are difficult to satisfy at the edge of a tokamak. In contrast, the analysis of the present paper is valid for arbitrary aspect ratio and for collisionalities typical of high-density tokamak discharges. By taking the large-aspect-ratio limit of these results, similarities with the corresponding results in the plateau regime[3] become apparent and the underlying physics is illuminated.

The paper is organized as follows. In Sec. II we derive the modified parallel momentum constraint and give the solution of the ion kinetic equation. In Sec. III we calculate the modification of the ion flow caused by the neutral diffusion and anomalous processes. Finally, in Sec. IV we summarize our results.

II. PARALLEL MOMENTUM CONSTRAINT

When ions and neutrals are coupled by charge exchange, it is often convenient to employ the equation obtained by adding the ion and neutral kinetic equations. In Ref. 3 it was shown that the retention of neutral diffusion due to charge exchange for ion charge $Z=1$ leads to the following ion plus neutral gyroaveraged kinetic equation in the short mean-free path regime:

$$(v_\parallel \hat{\mathbf{n}}+\mathbf{v}_d)\cdot\nabla[(1+\eta)\overline{f_i}]+\langle\delta\mathbf{v}_E\cdot\nabla[(1+\eta)\delta\overline{f_i}]\rangle_f + \frac{e}{M}E_* v_\parallel \frac{\partial\overline{f_i}}{\partial E}-C_{ii}(\overline{f_i})=\nabla\cdot\left[\tau\nabla\cdot\left(\frac{n_n}{n_i}\langle\mathbf{vv}\rangle_\varphi\overline{f_i}\right)\right], \quad (1)$$

where the gradients are taken at constant energy $E=v^2/2$ and magnetic moment $\mu=v_\perp^2/2B$. Here $\overline{f_i}$ is the gyroaveraged ion distribution function for the ions of density n_i, n_n is the neutral density, $\tau=1/n_i(\langle\sigma v\rangle_x+\langle\sigma v\rangle_z)$ with $\langle\sigma v\rangle_x$ and $\langle\sigma v\rangle_z$ the charge exchange and ionization rate constants, $\eta\equiv n_n\langle\sigma v\rangle_x\tau$, and $\delta\overline{f_i}$ and $\delta\overline{v}_E$ are the turbulent, fluctuating portions of f_i and the $\mathbf{E}\times\mathbf{B}$ drift. The average over turbulent fluctuations is denoted by $\langle\cdots\rangle_f$, while $\langle\cdots\rangle_\varphi$ represents a gyroaverage, so that $\langle\mathbf{vv}\rangle_\varphi=(v_\perp^2/2)(\mathbf{I}-\hat{\mathbf{n}}\hat{\mathbf{n}})+v_\parallel^2\hat{\mathbf{n}}\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}=\mathbf{B}/B$ and $B=|\mathbf{B}|$. The drift velocity is $\mathbf{v}_d=(c/B^2)\mathbf{E}\times\mathbf{B}+(1+\eta)(\mu/2\Omega)\hat{\mathbf{n}}\times\nabla B+(1+\eta)(v_\parallel^2/\Omega)\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}})$, where $\Omega=eB/Mc$ and M is the ion mass. The ion–ion collision operator is denoted by C_{ii} with the effect of ion–electron

[a)]Electronic mail: tunde@elmagn.chalmers.se collisions retained by defining $E_{*}=E_{\|}-(F_{\|}/en_e)$ with $F_{\|}=-M\int d^3v v_{\|}\overline{C_{ie}}$ and n_e the electron density. We will assume that the neutral density is much smaller than the ion density, and neglect the n_n/n_i corrections in Eq. (1), including $\eta\sim n_n/n_i$. Note that Eq. (1) has been obtained under the assumptions of a short neutral mean-free path and $\langle\sigma v\rangle_x \gtrsim \langle\sigma v\rangle_z$, implying approximately equal neutral and ion temperatures ($T_n\simeq T_i$).

The plasma flow velocity is determined by the parallel momentum constraint equation obtained by taking the $M\mathbf{v}\cdot\mathbf{B}=MBv_{\|}$ moment of the sum of Eq. (1) and the kinetic equations for the electrons, and performing a flux-surface average. If the electron contribution to the fluctuating term is neglected as small in the mass ratio, this equation is

$$\langle (p_{\|}-p_{\perp})\nabla_{\|}B\rangle = \langle(\nabla\cdot\mathbf{\Pi}_n)\cdot\mathbf{B}\rangle + M\left\langle\left\langle B\,\delta\mathbf{v}_E\cdot\nabla\int d^3v v_{\|}\overline{\delta f_i}\right\rangle_f\right\rangle, \quad (2)$$

where $\langle\cdots\rangle$ denotes flux-surface average and

$$\langle(\nabla\cdot\mathbf{\Pi}_n)\cdot\mathbf{B}\rangle \equiv -\left\langle MB\int d^3v v_{\|}\nabla\cdot[\tau\nabla\cdot(n_n/n_i\langle\mathbf{vv}\rangle_{\varphi}\overline{f_i})]\right\rangle. \quad (3)$$

In the Pfirsch–Schlüter regime, we need only retain the diagonal contributions to the ion stress tensor and may neglect all nonisotropic corrections to the electron stress tensor. Therefore, in Eq. (2) the ions dominate the $p_{\|}=M\int d^3v\overline{f_i}v_{\|}^2$ and $p_{\perp}=M\int d^3v\overline{f_i}v_{\perp}^2/2$ terms. The neutral viscosity term may be approximated as

$$\langle(\nabla\cdot\mathbf{\Pi}_n)\cdot\mathbf{B}\rangle \simeq -\left\langle\nabla\cdot\left[MB\tau\frac{n_n}{n_i}\nabla\cdot\left(\int d^3v\,\mathbf{vv}v_{\|}\overline{f_i}\right)\right]\right\rangle, \quad (4)$$

by assuming that variations in the magnetic field occur on a scale length that is much longer than the variation in the plasma and neutral densities and temperatures. We also neglect variations of B in the fluctuating term in Eq. (2).

The last term in Eq. (2) describes the anomalous transport of parallel momentum, and is retained to balance neutral diffusion. We approximate it simply as

$$\langle\delta\mathbf{v}_E\cdot\nabla\,\delta\overline{f_i}\rangle_f \simeq -\frac{\partial}{\partial r}\left(D\frac{\partial\overline{f_i}}{\partial r}\right), \quad (5)$$

where D is the anomalous diffusivity.

Also, we make the usual neoclassical assumption that $\overline{f_i}$ is a Maxwellian flux function to lowest order when written in terms of the total energy $v^2/2+e\Phi/M$, where Φ is the electrostatic potential. Recall that in neoclassical theory the poloidal variations in density, temperature, and Φ are usually assumed to be weak compared with the poloidal variation of the magnetic field $\mathbf{B}$. As a result, $\int d^3v v_{\|}\mathbf{v}_d\cdot\nabla\overline{f_i}=0$ to requisite order in the $Mv_{\|}B$ moment of Eq. (1).

Equation (2) is a generalization of the constraint that determines the parallel ion flow $V_{\|i}$ in the standard Pfirsch–Schlüter limit derived by Hazeltine.[6] It represents a balance between damping of the parallel flow by magnetic pumping[7] and the accumulation or loss of momentum due to neutral diffusion and anomalous flows.

To evaluate the various terms in (2), we first note that $p_{\|}-p_{\perp}=M\int d^3v\overline{f_i}P_2(\xi)v^2$. As a result, we need only determine the terms in $\overline{f_i}$ that are proportional to the Legendre polynomial $P_2(\xi)=(3\xi^2-1)/2$, where $\xi=v_{\|}/v$. For simplicity, we assume that the effect of the anomalous processes is isotropic in Eq. (1) [that is, D does not depend on pitch angle in Eq. (5)], so that they do not contribute any $P_2(\xi)$ terms and, therefore, do not introduce any additional anomalous terms into the parallel momentum constraint (2).

Except for the effect of neutral diffusion, the remaining terms in the gyroaveraged ion kinetic equation are the same as those in the corresponding equation in Ref. 6. Consequently, we may employ the results from Ref. 6 plus a $P_2(\xi)$ modification due to neutral diffusion obtained by solving

$$C_{ii}(h_n) = -\nabla\cdot\left[\tau\nabla\cdot\left(\frac{n_n}{3n_i}v^2P_2(\xi)(3\hat{\mathbf{n}}\hat{\mathbf{n}}-\mathbf{I})f_{\mathrm{M}}\right)\right]. \quad (6)$$

The details of a variational solution of this equation, obtained with a single variational parameter, are given in the Appendix. To the requisite order, the resulting expression for $\overline{f_i}$ is

$$\overline{f_i}=f_{\mathrm{Hazeltine}}+h_n+\text{isotropic terms}. \quad (7)$$

Here

$$f_{\mathrm{Hazeltine}}=f_{\mathrm{M}}+\frac{M}{T_i}V_{\|i}v_{\|}f_{\mathrm{M}}+h_T+h_U+h_{\rho}, \quad (8)$$

$$f_{\mathrm{M}}=n_i\left(\frac{M}{2\pi T_i}\right)^{3/2}\exp(-x^2), \quad (9)$$

$$h_T=-\frac{I}{15\Omega T_i}\frac{\partial T_i}{\partial\Psi}\left(1-\frac{B^2}{\langle B^2\rangle}\right)[x^2(1+2x^2)-20]v_{\|}f_{\mathrm{M}}, \quad (10)$$

$$h_U=-\tau_iP_2(\xi)Uf_{\mathrm{M}}(2.28x^2+1.056x^4)\nabla_{\|}\ln B, \quad (11)$$

$$h_{\rho}=1.2\frac{\tau_i cI}{eB^2}\frac{\partial T_i}{\partial\Psi}P_2(\xi)f_{\mathrm{M}}\left[\left(1+12.23\frac{B^2}{\langle B^2\rangle}\right)x^2 - \left(0.36+6.03\frac{B^2}{\langle B^2\rangle}\right)x^4\right]\nabla_{\|}B, \quad (12)$$

$$h_n=-\frac{5\tau_i x^2}{9n_iT_i}P_2(\xi)f_{\mathrm{M}}\nabla\cdot\left[\tau\nabla\cdot\left(\frac{n_nT_i^2}{M}(\mathbf{I}-3\hat{\mathbf{n}}\hat{\mathbf{n}})\right)\right], \quad (13)$$

where $x^2=Mv^2/2T_i$, and $V_{\|i}$ is the parallel ion velocity,

$$V_{\|i}=U-\frac{IT_i}{M\Omega}\left(\frac{1}{p_i}\frac{\partial p_i}{\partial\Psi}+\frac{e}{T_i}\frac{\partial\Phi}{\partial\Psi}\right), \quad (14)$$

with U/B an unknown flux function to be determined by the parallel momentum constraint, Eq. (2). In the preceding, $p_i=n_iT_i$, Ψ is the poloidal flux function, $I=RB_t$ with R the major radius and B_t the toroidal magnetic field, and

$$\tau_i=\frac{3T_i^{3/2}M^{1/2}}{4\sqrt{\pi}e^4n_i\ln\Lambda}, \quad (15)$$

is the ion–ion collision time with $\ln\Lambda$ the Coulomb logarithm.

The isotropic part of $\overline{f_i}$ is not needed for the calculation of the ion parallel flow since it does not contribute to the parallel momentum conservation equation. Also note that h_T is constructed such that $\int d^3v v_\parallel h_T=0$.

III. MODIFICATION OF THE ION FLOW

Having determined the ion distribution function, we can proceed to evaluate the various terms in the flux-surface-averaged momentum constraint, Eq. (2). Recalling that U/B must be a flux function, we find

$$\langle (p_\parallel - p_\perp)\nabla_\parallel B\rangle = -9 p_i \tau_i \langle (\nabla_\parallel B)^2\rangle \frac{V_{\parallel i}-V_{\parallel i}^{\rm neo}-V_n}{B} \tag{16}$$

and

$$\langle (\nabla\cdot\Pi_n)\cdot\mathbf{B}\rangle = -\frac{1}{V'}\frac{\partial}{\partial\Psi} \times V'\left\langle \tau\nabla\Psi\cdot\nabla\left[Bn_nT_i\left(V_{\parallel i}+\frac{2q_{\parallel i}}{5p_i}\right)\right]\right\rangle \equiv -9p_i\tau_i\langle(\nabla_\parallel B)^2\rangle\frac{V_\pi}{B}, \tag{17}$$

where we define the usual Pfirsch–Schlüter parallel flow velocity as

$$V_{\parallel i}^{\rm neo} = -\frac{IT_i}{M\Omega}\left[\frac{1}{p_i}\frac{\partial p_i}{\partial\Psi}+\frac{e}{T_i}\frac{\partial\Phi}{\partial\Psi}+\frac{1}{T_i}\frac{\partial T_i}{\partial\Psi} \times\left(1.8\frac{B^2}{\langle B^2\rangle}+0.05\frac{B^2\langle(\nabla_\parallel \ln B)^2\rangle}{\langle(\nabla_\parallel B)^2\rangle}\right)\right], \tag{18}$$

and the contributions to the flow from h_n and Π_n, respectively, as

$$V_n = -0.09\frac{B\langle\nabla\cdot[\tau(\nabla_\parallel B)\nabla(n_nT_i^2)]\rangle}{Mp_i\langle(\nabla_\parallel B)^2\rangle} = -\frac{0.09B}{Mp_i\langle(\nabla_\parallel B)^2\rangle V'}\frac{\partial}{\partial\Psi}V'\langle\tau(\nabla_\parallel B)\nabla\Psi\cdot\nabla(n_nT_i^2)\rangle, \tag{19}$$

and

$$V_\pi = \frac{0.11B}{p_i\tau_i\langle(\nabla_\parallel B)^2\rangle V'}\frac{\partial}{\partial\Psi} \times V'\left\langle\tau\nabla\Psi\cdot\nabla\left[Bn_nT_i\left(V_{\parallel i}+\frac{2q_{\parallel i}}{5p_i}\right)\right]\right\rangle. \tag{20}$$

Here, the parallel heat flux is

$$q_{\parallel i} = \int d^3v\left(\frac{Mv^2}{2}-\frac{5T}{2}\right)v_\parallel\overline{f_i} = -\frac{5Ip_i}{2M\Omega}\frac{\partial T_i}{\partial\Psi}\left(1-\frac{B^2}{\langle B^2\rangle}\right), \tag{21}$$

and $V'=\int d\theta/(\mathbf{B}\cdot\nabla\theta)$.

Defining the anomalous flow contribution as

$$V_a = -\frac{0.11MB}{p_i\tau_i\langle(\nabla_\parallel B)^2\rangle}\left\langle\left\langle B\,\delta\mathbf{v}_E\cdot\nabla\int d^3v v_\parallel\delta\overline{f_i}\right\rangle_f\right\rangle, \tag{22}$$

the full flux-surface-averaged parallel momentum constraint allows us to determine the flow U, or equivalently, the parallel ion flow $V_{\parallel i}$, to be

$$V_{\parallel i} = V_{\parallel i}^{\rm neo}+V_n+V_\pi+V_a, \tag{23}$$

which is the main result of this paper. The three new terms in Eq. (23) correspond to the contributions from h_n, $\nabla\cdot\Pi_n$ and the anomalous term, respectively. The poloidal flow velocity is

$$V_\theta = \frac{UB_\theta}{B}, \tag{24}$$

where U is related to $V_{\parallel i}$ by Eq. (14) and $B_\theta=\mathbf{B}\cdot\nabla\theta/|\nabla\theta|$.

Based on Eq. (5) we may estimate the anomalous term in Eq. (23) as

$$\left\langle\left\langle\delta\mathbf{v}_E\cdot\nabla\int d^3v v_\parallel\delta\overline{f_i}\right\rangle_f\right\rangle \sim -\frac{\partial}{\partial r}\left(D\frac{\partial(n_iV_{\parallel i})}{\partial r}\right). \tag{25}$$

The relative importance of the different contributions can be estimated by

$$\frac{V_\pi}{V_n}\sim\left(\frac{qR}{\epsilon\lambda_i}\right)\left(\frac{q\rho_i}{\epsilon W}\right), \tag{26}$$

$$\frac{V_\pi}{V_{\parallel i}^{\rm neo}}\sim\left(\frac{qR}{\epsilon W}\right)^2\left(\frac{n_n\lambda_x}{n_i\lambda_i}\right), \tag{27}$$

$$\frac{V_a}{V_{\parallel i}^{\rm neo}}\sim\left(\frac{qR}{\epsilon W}\right)^2\left(\frac{D}{v_{ti}\lambda_i}\right), \tag{28}$$

where $v_{ti}^2=2T_i/M$ is the ion thermal speed, $\lambda_i=v_{ti}\tau_i$ is the Coulomb mean-free path, $\lambda_x=v_{ti}/(n_i\langle\sigma v\rangle_x)$ is the neutral charge exchange mean-free path, $\epsilon=r/R$ is the inverse aspect ratio, and W is the local radial density scale length.

Note that the neutrals affect the plasma flow in two different ways, represented by the terms V_n and V_π. The term V_π is caused by the additional parallel viscosity provided by the neutral population, while V_n reflects a direct modification of the ion distribution function by the ion–neutral interaction. The latter is comparable to the neoclassical contribution if $h_n\sim h_\rho$, which gives $(n_nR\Omega)/(n_iW\nu_x)\sim 1$.

A. Large aspect ratio limit ($\epsilon\ll 1$)

Keeping both the h_n and $\langle(\nabla\cdot\Pi_n)\cdot\mathbf{B}\rangle$ contributions in the constraint equation corresponds to the ordering $V_\pi\sim V_n$ or

$$\epsilon\sim\frac{qR}{\lambda_i}\frac{q\rho_i}{W\epsilon}. \tag{29}$$

The neglect of the poloidal variation of plasma density, ion temperature, and electrostatic potential compared to the magnetic field variation in the Pfirsch–Schlüter regime requires that $(qR/\lambda_i)(q\rho_i/W\epsilon)\ll 1$. Consequently, both V_π and V_n must be retained when the aspect ratio is large.

Comparing Eq. (23) with the corresponding result in the plateau regime,[3] we note that the term V_n modifying the parallel ion flow $V_{\parallel i}$ was not obtained, and the factor corresponding to V_π was smaller by λ_i/qR. Recall that the new flow term V_n arises from h_n, and therefore represents the direct effect of the neutrals on the ion distribution. Such an effect is not included in the plateau calculation since $\epsilon \ll q\rho_i/\epsilon W \ll 1$ must be assumed. In addition, the $q_{\parallel i}$ term in V_π, which is associated with the parallel heat flux and is due to the temperature gradient term h_T, does not arise in the plateau regime.

B. Order unity aspect ratio $[\epsilon = O(1)]$

If ϵ is of order unity, then $V_n \gg V_\pi$, since the assumption that the poloidal variation of the B field dominates over the density and temperature variations requires

$$\frac{\lambda_i}{qR}\frac{W}{q\rho_i} \gg 1. \tag{30}$$

In this case, we may neglect the neutral diffusion term $\langle(\nabla\cdot\mathbf{\Pi}_n)\cdot\mathbf{B}\rangle$ in the parallel momentum constraint. We continue to retain anomalous effects by assuming $D \gg n_n v_{ti}\lambda_x/n_i$. As a result, the $\epsilon = O(1)$ parallel momentum constraint (2) leads to the following expression for the parallel ion flow:

$$V_{\parallel i} \simeq V_{\parallel i}^{\rm neo} + V_n + V_a, \tag{31}$$

so that derivatives of $V_{\parallel i}$ only enter in the last term.

C. Estimates

To estimate whether the modification of the parallel ion flow can be significant, let us evaluate Eqs. (26)–(28) using the following set of parameters: safety factor $q=3$, inverse aspect ratio $\epsilon = 1/3$, the neutral to ion density ratio $n_n/n_i = 0.5\times 10^{-3}$, the ion thermal velocity $v_{ti} = 10^7$ cm/s, the characteristic radial scale length $W = 2$ cm, major radius $R = 1$ m, charge exchange mean-free path $\lambda = 1$ cm, anomalous diffusivity $D = 1\ \mathrm{m}^2/\mathrm{s}$, Coulomb mean-free path $\lambda_i = 1$ m, and gyroradius $\rho_i = 0.02$ cm. These parameters correspond to an ion temperature of $T_i = 100$ eV, magnetic field $B = 5$ T, ion density $n_i = 10^{14}\ \mathrm{cm}^{-3}$ and an ion–neutral cross section of $\sigma_x = 6\times 10^{-15}\ \mathrm{cm}^2$. We then find that $V_\pi/V_n \simeq 1$ and $V_\pi/V_{\parallel i}^{\rm neo} \simeq V_a/V_{\parallel i}^{\rm neo} \simeq 2$. Consequently, all three modifications of the parallel ion flow, V_n, V_π, and V_a, can be important.

IV. CONCLUSIONS

We have calculated the effect of neutral diffusion and anomalous transport on the parallel and poloidal ion flows in the Pfirsch–Schlüter regime for arbitrary aspect ratios. The result is summarized by Eq. (23), and shows that the ion flow velocity is equal to the sum of four terms, each governed by different physics. Neutral atoms affect the flow velocity in two ways: (i) by directly providing additional parallel viscous damping; and (ii) by collisional modification of the ion distribution function and its associated parallel viscosity. Only the latter is important if $\epsilon = O(1)$. The modification of the flow in the Pfirsch–Schlüter regime is a factor of qR/λ_i larger than in the plateau regime, and is significant even for neutral densities less than a thousandth of the ion density. This general conclusion is independent of the aspect ratio, and may be of particular importance in spherical tokamaks, where experimental evidence suggests that neutral–plasma interaction can be particularly strong. Since the neutral particle fraction in most tokamak edge plasmas is no less than 10^{-4}, it appears likely that neutral atoms generally have a significant influence on the plasma flow. Our main result (23) indicates that the amount of ion flow shear at the edge is set by a delicate balance between neutral and anomalous viscous effects, which may then self-consistently suppress the turbulence and determine the steady-state fluctuation level.[1]

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APPENDIX A: VARIATIONAL TREATMENT OF THE NEUTRAL DIFFUSION

We desire to solve the following Spitzer problem:

$$C_l(g) = -\nabla\cdot\left[\frac{1}{n_i\langle\sigma v\rangle_x}\nabla\cdot\left(\frac{n_n}{n_i}v^2P_2(\xi)(\hat{\mathbf{n}}\hat{\mathbf{n}}-\mathbf{I}/3)\right)f_{\rm M}\right] \equiv \beta f_{\rm M},$$

where C_l denotes the self-adjoint linearized ion–ion collision operator. We define the functional

$$\Lambda = \int d^3v\, hC_l(hf_{\rm M}) - 2\int d^3v\, h\beta f_{\rm M},$$

which is variational ($\delta\Lambda = 0$ for $h = g$) and maximal $\delta^2\Lambda \leqslant 0$. Assuming a trial function $h = cv^2P_2(\xi)$ we can calculate the two terms in Λ to be[8]

$$\int d^3v\, hC_l(hf_{\rm M}) = -\frac{18p_iT_ic^2}{5\tau_iM^2}$$

and

$$\int d^3v\, h\beta f_{\rm M} = -\nabla\cdot\left[\frac{1}{n_i\langle\sigma v\rangle_x}\nabla\cdot\left(\frac{n_nT_i^2c}{M}(3\hat{\mathbf{n}}\hat{\mathbf{n}}-\mathbf{I})\right)\right].$$

Minimizing the functional with respect to the variational parameter $c(\partial\Lambda/\partial c = 0)$, we obtain c, and, thereby, the variational solution

$$h = -\frac{5\tau_iM^2}{18n_iT_i^2}v^2P_2(\xi)f_{\rm M} \times\nabla\cdot\left[\frac{1}{n_i\langle\sigma v\rangle_x}\nabla\cdot\left(\frac{n_nT_i^2}{M^2}(\mathbf{I}-3\hat{\mathbf{n}}\hat{\mathbf{n}})\right)\right].$$

[1] K. H. Burrell, Phys. Plasmas **4**, 1499 (1997).
[2] W. L. Rowan, A. G. Meigs, E. R. Solano, P. M. Valanju, M. D. Calvin, and R. D. Hazeltine, Phys. Fluids B **5**, 2485 (1993).
[3] T. Fülöp, P. J. Catto, and P. Helander, Phys. Plasmas **5**, 3398 (1998).
[4] P. M. Valanju, M. D. Calvin, R. D. Hazeltine, and E. R. Solano, Phys. Fluids B **4**, 2675 (1992).
[5] P. Monier-Garbet, K. H. Burrell, F. L. Hinton, J. Kim, X. Garbet, and R.

J. Groebner, Nucl. Fusion **37**, 403 (1997); M. Coronado and J. N. Talmadge, Phys. Fluids B **5**, 1200 (1993).
[6]R. D. Hazeltine, Phys. Fluids **17**, 961 (1974), with the following corrections: (i) 0.27 replaced by 0.05 in Eqs. (60) and (61); (ii) 3.1 replaced with 2.9 in Eq. (67); (iii) -2.1 replaced by -1.9 in the $\Delta \ll 1$ form of Eq. (69); (iv) the slash removed in S_1^0 between Eqs. (51) and (52) so it reads as $S_1^0 = x^3 + 2x^5$; (v) an extra power of mass removed in Eq. (77); and (vi) the last term in Eq. (82) should be $3.1 n \partial^2 T/\partial t \partial r$.
[7]A. B. Hassam and R. M. Kulsrud, Phys. Fluids **21**, 2271 (1978).
[8]P. J. Catto, I. B. Bernstein, and M. Tessarotto, Phys. Fluids **30**, 2784 (1987).