Kinetic modelling of runaway in plasmas

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Abstract
The phenomenon of runaway occurs in plasmas in the presence of a strong electric field, which overcomes the collisional friction acting on the charged particles moving through the plasma. A subpopulation of particles can then be accelerated to energies significantly higher than the thermal energy. Such events are observed in both laboratory and space plasmas, and are of great importance in fusion-energy research, where highly energetic runaway electrons can damage the plasma-facing components of the reactor.

In this thesis, a series of papers are presented which investigate various aspects of runaway dynamics. The emission of synchrotron and bremsstrahlung radiation are important energy-loss mechanisms for relativistic runaway electrons. Photons emitted in bremsstrahlung radiation often have energy comparable to the energetic electrons, and we therefore use a Boltzmann transport equation in order to describe their effect on the electron motion. This treatment reveals that electrons can reach significantly higher energies than previously thought. In comparison, synchrotron radiation has lower frequency, and is well described by the classical electromagnetic radiation-reaction force. This loss mechanism, often dominant in laboratory plasmas, significantly alters the electron dynamics, and is found to produce non-monotonic features in the runaway tail.

A study is also presented of the related phenomenon of ion runaway acceleration, which differs from electron runaway due to their larger mass. Renewed interest in this topic has been sparked after recent observations of fast ions in various experiments. Finally a new method is explored to treat the non-linear Fokker-Planck equation which is commonly used to describe the collisional dynamics in a plasma. The new method is appealing for its physically intuitive description and analytic simplicity.

Keywords: plasma, runaway, Boltzmann equation, Fokker-Planck equation, bremsstrahlung, synchrotron radiation
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*Effect of bremsstrahlung radiation emission on fast electrons in plasmas,*
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B. A. Stahl, O. Embréus, G. Papp, M. Landreman and T. Fülöp,
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# Contents

Abstract ........................... i  
Publications .......................... iii  
Acknowledgements .................... vi  

1 Introduction .......................... 1  
  1.1 Runaway generation ................ 3  
  1.2 Runaway in tokamaks ................ 4  
  1.3 Ion runaway ......................... 6  

2 The kinetic equation .................. 7  
  2.1 BBGKY hierarchy and the kinetic equation .......... 9  
  2.2 The Boltzmann collision operator .......... 11  
  2.3 The Fokker-Planck collision operator .......... 16  
  2.4 Synchrotron radiation reaction .......... 18  
  2.5 CODE ............................. 20  

3 Bremsstrahlung ....................... 23  
  3.1 Screening .......................... 26  
  3.2 Low-energy photon contribution .......... 27  
  3.3 Lower limit in photon energy .......... 29  

4 Ion runaway .......................... 39  
  4.1 Ion friction-force estimates ................ 42  

5 Conclusions and outlook .............. 45  

Bibliography .......................... 49
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Chapter 1

Introduction

A plasma is an ionized gas, sufficiently hot that the electrons have detached from the atoms that carried them. Because it consists of free charges, rather than neutral atoms, the plasma behaves differently to the familiar gases and fluids encountered in everyday life. Indeed, the addition of electric and magnetic forces between the particles creates a rich interplay, allowing a wide range of strange and wonderful phenomena to occur. Some of these are well-known to most: lightning, electric sparks, fluorescent lamps, the Sun and the stars, and even the *aurora borealis* – the northern lights – are examples of plasmas. In fact, a majority of the visible matter in the universe is in the plasma state. The study of plasmas is a huge field of research, ranging from academic contemplations in astrophysical research and space physics, through fusion-energy research to various industrial and medical applications.

Runaway is a phenomenon which occurs in any plasma in the presence of a sufficiently strong electric field. It is a process related to dielectric breakdown, which occurs when electric sparks are created. Runaway breakdown occurs in laboratory plasmas, such as those in tokamak fusion devices [1], as well as in lightning discharges during thunderstorms [2], and in astrophysical plasmas, such as solar flares [3]. In these scenarios, a subpopulation of particles — typically electrons, which are lightest — are accelerated by the applied field to energies significantly higher than the thermal energy, at which point they start emitting radiation.

The phenomenon of runaway can be understood by considering in detail the frictional drag force due to collisions which acts on a charged particle moving through a plasma which is near thermodynamic equilibrium. The friction is a non-monotonic function of speed: at low speed,
the drag steadily grows in magnitude as the speed increases; however, above the thermal speed of the particles, the drag force will instead decrease in magnitude as the speed increases further. In the absence of an electric field, the friction force on the thermal particles will be balanced by velocity-space diffusion induced by collisions, which tends to increase the width of the velocity distribution. An equilibrium between friction and diffusion is reached when the distribution takes the Maxwellian form, \( f_M = n(m/2\pi T)^{3/2} \exp(-mv^2/2T) \), where \( m, n \) and \( T \) are the mass, number density and temperature (in energy units, throughout this work) of the species, respectively, and \( v \) is the speed.

In the presence of an electric field which acts to accelerate charged particles, an electron with sufficiently high initial speed will experience an unbounded acceleration to highly relativistic energies, where the electrons move at nearly the speed of light. At these energies competing physical effects become important, such as radiation losses caused by the rapidly accelerated motion experienced by the particles when moving in electromagnetic fields (leading to synchrotron radiation) or in collisions (causing bremsstrahlung emission). Figure 1.1 schematically illustrates the forces which act on a runaway electron.

![Figure 1.1: A schematic view of the speed-dependent force acting on a particle in a plasma, showing friction due to collisions and radiation (solid, black) and acceleration by an electric field (dashed, red). Not to scale (the speed where radiation losses become important can be thousands of times larger than the thermal speed).](image-url)
1.1 Runaway generation

Historically, the runaway phenomenon in plasmas was first discussed by Dreicer [4] in 1959. He considered the total friction force between two Maxwellian particle species moving uniformly with a given speed relative to each other. When accelerated by a sufficiently strong electric field, greater than approximately 43\% of the so-called Dreicer field $E_D = n_e \ln \Lambda e^3/(4\pi\varepsilon_0^2 T_e)$, the electric field overcomes the maximum frictional force, and he concluded that the particles would “run away” towards infinite energy (given infinite time). The Coulomb logarithm $\ln \Lambda$ is a plasma parameter which typically takes values 10-20 in the applications we consider [5].

In 1964, Kruskal and Bernstein [6] rigorously treated the runaway problem with an analytic solution to the kinetic equation (although using a simplified model for collisions). They solved the kinetic equation with an asymptotic technique, matching approximate solutions across five regions in momentum space, thereby obtaining expressions for the shape of the velocity distribution of runaway electrons and the rate at which new runaways are generated (here called the runaway growth rate, or runaway rate). It was found that all electrons moving with a velocity above a critical velocity $v_c$, defined approximately as that velocity above which the electric field becomes stronger than friction, will run away towards infinite energy. In addition, diffusion would supply the runaway region ($v > v_c$) with new particles from the bulk at a constant rate. This mechanism of runaway generation is referred to as primary generation, or Dreicer generation.

Early work on runaways considered primarily the initial generation of runaways at relatively low, non-relativistic speeds. A full description of electron runaway requires the use of a relativistic kinetic equation, a scenario which was first analyzed by Connor and Hastie [7] in 1975. They extended the method of Kruskal and Bernstein to account also for relativistic effects. Unlike the case of non-relativistic runaway, where the frictional force appears to tend towards zero for large speeds, the friction in the relativistic model attains a minimum value, corresponding to a critical electric field $eE_c = n_e \ln \Lambda e^4/(4\pi\varepsilon_0^2 m_e c^2)$. This is smaller than the Dreicer field by the factor $T/m_e c^2$. It is found that the runaway growth rate $\Gamma = \partial n_{RE}/\partial t$ is exponentially sensitive to the electric field, scaling approximately as $\Gamma \propto \exp(-\lambda E_D/4E)$, with a (generally) small correction factor $\lambda(E)$ which forces the growth rate to zero as $E \to E_c$.

In a seminal paper by Rosenbluth and Putvinski [8] in 1997, the ef-
fect of large-angle collisions on the runaway rate was detailed. Runaway generation by large-angle collisions (also called knock-on or close collisions) is possible when runaway electrons already exist in the plasma. Then, in a single large-momentum-transfer collision event, an initially slow electron may obtain a velocity great enough to enter the runaway region where friction is smaller than the electric force. Thus, an initial runaway particle can become two after undergoing a knock-on collision, which can become four after further knock-on collisions, and so on. Therefore, the runaway rate due to knock-on collisions is proportional to the number of runaways already present, which causes an exponential growth of the runaway population and hence is referred to as a runaway avalanche. Large-angle collisions in plasmas will generally have a smaller effect on the evolution of the distribution than small-angle collisions by a factor of the inverse Coulomb logarithm $1/\ln \Lambda$. Since Dreicer generation is exponentially sensitive to the electric field, however, it will be completely negligible for small enough $E/E_D$. Because of this large-angle collisions can sometimes dominate the runaway rate even though they give a “small” modification to the equation (in the sense of being of order $1/\ln \Lambda$), with a contribution to the runaway rate that scales as $\Gamma \propto n_{RE}(E/E_c - 1)$.

## 1.2 Runaway in tokamaks

Runaways are of particular interest in magnetic-fusion research, where they pose a great threat to the successful operation of tokamaks [9]. These are a promising concept for fusion-energy reactors, which confine the plasma by magnetic fields and heat it to several hundred million K. A magnetic field is partially generated by driving a strong current of several MA through the plasma, with the downside that this is then available for conversion into a runaway-carried current. The mechanism for this conversion is the runaway breakdown. This typically occurs during so-called disruptions, which are sudden unintentional events where heat confinement is lost. During these disruptions the plasma loses its energy and cools rapidly on a timescale of milliseconds [10], sometimes to one-thousandth of its original temperature. The temperature reduction is associated with a decrease in the electrical conductivity of the plasma, which causes the large plasma current to rapidly decay. By the induction equation (or Lenz’s law), this induces an electric field in order to maintain the current, finally enabling runaway breakdown to occur.
It can be shown that avalanche runaway multiplication often dominates the runaway generation, although initiation of the avalanche requires the presence of an initial seed population of runaways. This can be provided either by Dreicer generation, as described above, or by so-called hot-tail generation [11, 12, 13, 14]. This third generation mechanism is enabled by the rapid temperature change that occurs during disruptions. If the cooling is sufficiently fast, the fastest particles in the tail of the thermal distribution (which experience a weaker drag force) will maintain their initial energy. During the cooling their velocities may at some point be greater than the critical velocity for runaway generation, and thus they can become runaways.

It can be shown that, if there is an initial seed population $n_{RE,0}$ of runaways, avalanche multiplication will increase this number to approximately $n_{RE} \sim \exp (2.5I[MA]) n_{RE,0}$ before the electric field has decayed [9], where $I[MA]$ is the original plasma current in MA. While the multiplication factor is fairly small in present-day experiments (of order $10^4$ [9] in the JET tokamak [15], the biggest current experiment), in future tokamaks such as the international collaboration ITER [16] this implies a devastating number of $10^{16}$ or greater [9]. Because of this immense number, runaway-electron dynamics and disruption mitigation is a field of active study. Recent reviews can be found in Refs. [17, 18, 19, 20].

The qualitative features of the basic runaway phenomenon in plasmas can thus be summarized as:

- Runaway is only possible for electric fields exceeding the critical field, $E > E_c$.
- Primary (Dreicer) runaway generation is exponentially sensitive to electric field, and only gives an appreciable growth rate when $E \gtrsim 0.01E_D$.
- Secondary (avalanche) runaway generation depends weakly on electric field and is caused by knock-on collisions, requiring a significant fraction of runaways to already be present in the plasma.
- Hot-tail runaway generation is caused by a rapid temperature drop, and describes the conversion of previously thermal electrons into runaways.

At highly relativistic (multi-MeV) energies, additional effects such as radiation losses become important for the dynamics of the fast electrons. These effects only weakly impact runaway generation, however, which
typically occurs at low velocities. In Papers A and C we have investigated in detail the effect of the radiation losses by synchrotron radiation and bremsstrahlung emission. A significant fraction of the plasma energy can be emitted in the form of such radiation, and these effects will often have a strong impact on the motion of the electrons.

1.3 Ion runaway

Runaway acceleration of ions was first invoked in order to explain experimental observations at the Zeta device [21] in 1959. In 1972, Furth and Rutherford [22] used an asymptotic technique similar to that used for electron runaway in order to obtain an analytic solution of the ion drift-kinetic equation. Their treatment provided only limited information about the runaway growth rate in most scenarios, however, due to the more complicated structure of the ion kinetic equation. A limited time-dependent solution of the ion kinetic equation was more recently developed in order to explain observations at the Mega Ampere Spherical Tokamak [23, 24, 25]. Simpler test-particle methods have also been used to study the ion runaway phenomenon in astrophysical contexts [26]. The lack of widely applicable analytic results has motivated a numerical study of the ion drift-kinetic equation, which is presented in Paper D. Experimentally observed ion acceleration in the Madison Symmetric Torus reversed-field pinch has lead to recent work where similar methods have been employed [27, 28].

Outline

Chapter 2 contains an introduction to the kinetic theory of plasmas, which describes the phase-space dynamics of charged particles. The theory presented here covers the physics required to understand the basic runaway phenomenon, but is also the foundation upon which further extensions of the theory can be developed. In chapter 3, we further develop the theory by describing a model for the effect of bremsstrahlung emission based on the Boltzmann collision operator, and its validity is investigated in detail. The theory for ion runaway requires a modified treatment compared to electron runaway; this theory is briefly summarized in chapter 4, and here we also derive improved analytic formulas for the runaway velocity and critical electric field. Finally the work is concluded in chapter 5.
Chapter 2

The kinetic equation

A detailed study of runaway particles requires the resolution of their momentum-space structure, accounting for the randomizing collisions in an accurate way. This is achieved using a kinetic equation, which provides a full description of the time evolution of the distribution function $f_a(t, x, p)$ of a particle species $a$, where $t$ is time, $x$ is the particle position, $p = m_a v / \sqrt{1 - v^2 / c^2}$ is the momentum and $v$ is the velocity. The distribution function is the particle density function in phase space, defined such that $n_a(t, x) = \int dp \ f_a(t, x, p)$ is the number density, and $N_a(t) = \int dx n_a(t, x)$ is the total number of particles of species $a$. In the absence of collisions, the distribution function describes particles moving along trajectories $x = x(t)$ and $p = p(t)$, governed by the equations of motion for a charged particle

\[
\frac{dx}{dt} = v,
\]
\[
\frac{dp}{dt} = q_a \left( E + v \times B \right),
\]

where $q_a$ is the charge. The continuity equation in phase space is [29]

\[
0 = \frac{df_a}{dt} = \frac{\partial f_a}{\partial t} + v \cdot \frac{\partial f_a}{\partial x} + q_a \left( E + v \times B \right) \cdot \frac{\partial f_a}{\partial p},
\]

(2.1)
where the electric and magnetic fields are given by the charge and current
distribution of the plasma according to Maxwell’s equations,

\[
\mathbf{E}(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \mathrm{d}\mathbf{x}' \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \rho(t, \mathbf{x}'),
\]
\[
\mathbf{B}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int \mathrm{d}\mathbf{x}' \mathbf{j}(t, \mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3},
\]

and the charge and current are in turn determined by the distribution functions,

\[
\rho(t, \mathbf{x}) = \sum_b e_b \int \mathrm{d}\mathbf{p} f_b(t, \mathbf{x}, \mathbf{p}),
\]
\[
\mathbf{j}(t, \mathbf{x}) = \sum_b e_b \int \mathrm{d}\mathbf{p} \mathbf{v} f_b(t, \mathbf{x}, \mathbf{p}),
\]

with the sum taken over all particle species \(b\) present in the plasma. In order to obtain a useful kinetic equation, Eq. (2.1) needs to be ensemble-averaged over macroscopically equivalent systems. The distribution function will then be a smooth function, but the microscopic interactions between the discrete particles in the plasma will need to be accounted for by the addition of a new term [30], which is called the collision operator \(C\), or the collision integral (as it generally takes the form of an integral operator). The kinetic equation then takes the form

\[
\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}} + q_a (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_a}{\partial \mathbf{p}} = \sum_b C_{ab}\{f_a, f_b\}, \quad (2.2)
\]

where the electromagnetic fields denote the macroscopic fields, not including fluctuations caused by individual particles which are instead captured by the collision operator \(C\). We shall focus on the simplest physical scenario that exhibits the runaway phenomenon: an infinite homogeneous plasma with an electric field in a constant direction. In this case we can suppress the space variables and write \(f_a = f_a(t, \mathbf{p})\), and introduce a spherical momentum coordinate system \((p, \theta, \varphi)\), where the azimuthal angle \(\varphi\) is referred to as the gyroangle, the longitudinal angle \(\theta\) is referred to as the pitch-angle, and we shall mostly use its cosine \(\xi = \cos \theta\). The kinetic equation then becomes

\[
\frac{\partial f_a}{\partial t} + q_a E \left(\xi \frac{\partial f_a}{\partial p} + \frac{1 - \xi^2}{p} \frac{\partial f_a}{\partial \xi}\right) = \sum_b C_{ab}\{f_a, f_b\}. \quad (2.3)
\]
This equation allows us to study the effects of various contributions in the collision operator $C$ on the dynamics of the runaway particles. We do not at present take an interest in the dynamical evolution of the electric field but instead assume it to be some prescribed external field $E = E(t)$.

An essential part of the description of runaway electrons is the collision operator. This term in the kinetic equation describes the effect of microscopic particle-particle interactions, in contrast to the macroscopic interactions with the electromagnetic field set up by the charge distribution in the plasma. The collisions drive the particle distributions towards thermal equilibrium by always increasing entropy in the system, and this is the restoring effect which needs to be overcome by the electric field in order to generate runaway particles. Therefore the details of the collision operator can be expected to strongly influence the description of the runaway process.

In this chapter we will provide a detailed discussion of the collision operator, revealing a unified picture of small-angle collisions, knock-on collisions and bremsstrahlung radiation in the same framework. We shall begin by investigating in more detail how the collision operator can be obtained.

2.1 BBGKY hierarchy and the kinetic equation

A systematic framework for obtaining kinetic equations was initially developed by Bogolyubov, Born, Green, Kirkwood and Yvon (BBGKY) [31, 32, 33, 34, 35]. The starting point of their analysis is the Liouville theorem [36], which deterministically describes the time evolution of an $N$-body system according to Hamiltonian mechanics. The system is fully described by the phase space density function $f_N(t, x_1, p_1, ..., x_N, p_N)$ giving the location and momentum of all its constituents. A kinetic equation describes the time evolution of the distribution function, which is defined as $f(t, x, p) = \int dx_2 dp_2 \cdots dx_N dp_N f_N(t, x, p, x_2, p_2, ..., x_N, p_N)$. Note that, while this definition appears to single out the particle with subscript 1 as special, the particles described by the phase-space density function are identical, and hence it is symmetric in all indices. That is, non-identical particle species are each described by their own function.

The Liouville equation [37] for a species interacting pair-wise with a central potential $V_{ij} = V(|x_i - x_j|)$, with the force on particle $i$ being $F_i = -\sum_{(j \neq i)=1}^N \partial V_{ij} / \partial x_i$ (for simplicity assuming a single species and
no magnetic interaction, which would require a generalized form of the potential), is given by

\[ \frac{\partial f_N}{\partial t} + \sum_{i=1}^{N} \frac{p_i}{m_a} \cdot \frac{\partial f_N}{\partial x_i} - \sum_{i=1}^{N} \sum_{(j \neq i)=1}^{N} \frac{\partial V_{ij}}{\partial x_i} \cdot \frac{\partial f_N}{\partial p_i} = 0. \]

By integrating over all but \( s \) particle coordinates, a reduced phase space density, or the \( s \)-particle correlation function, can be defined as \( f_s = \int dx_{s+1} dp_{s+1} \cdots dx_N dp_N f_N \). Here, \( s = 1 \) gives the distribution function in which we are most interested, and \( s = N \) returns the full \( N \)-particle phase space density. When performing such an integration over the Liouville equation, an equation for the time-evolution of the reduced distribution function is obtained; however, the equation for \( \frac{\partial f_s}{\partial t} \) invariably contains \( f_{s+1} \) due to the pair-wise interaction term. Thus, the time-evolution of the distribution function depends on the two-particle correlation function \( f_2 \), which in turn is affected by \( f_3 \), and so on. This set of coupled partial differential equations is called the BBGKY hierarchy. A systematic approximation scheme to close this set of equations was developed by Frieman [38], Sandri [39] and collaborators, which takes the form of a perturbation expansion in two parameters \( \mu \) and \( \eta \). These appear naturally when normalizing the equation to characteristic values of particle separation \( r_0 \), velocities \( v_0 \) and interaction strength \( V_0 \), and are given by

\[ \mu = \frac{1}{nr_0^3}, \]
\[ \eta = \frac{V_0}{mv_0^2} \sim \frac{e^2}{4\pi \varepsilon_0 Tr_0}. \]

Here \( 1/\mu \) is the number of particles in the interaction region (defined by a characteristic range \( r_0 \)), and \( \eta \) is a measure of the strength of the interaction (described by the potential function \( V_0 \)) compared to the kinetic energy. There are three domains of primary interest [40, 41] which can be described as (1) “dilute, short-range”, (2) “weak coupling” (small momentum transfer) and (3) “long-range”. These, respectively, correspond to the choices (with \( \epsilon \) a small expansion parameter)

\[ (1) \quad \mu = \mathcal{O}(\epsilon^{-1}), \quad \eta = \mathcal{O}(1), \]
\[ (2) \quad \mu = \mathcal{O}(1), \quad \eta = \mathcal{O}(\epsilon), \]
\[ (3) \quad \mu = \mathcal{O}(\epsilon), \quad \eta = \mathcal{O}(\epsilon). \]
These lead, in turn, to the so-called Boltzmann equation, the Fokker-Planck equation and the Balescu-Lenard (or Bogolyubov-Lenard-Balescu) equation. Plasmas are particularly pathological, as no specific ordering applies to the entire phase space. The long-range Coulomb interaction allows for collisions where any of the orderings may apply, depending on the impact parameter.

An analysis shows that the Balescu-Lenard operator takes a similar form to the Fokker-Planck operator, but where the dielectric constant of the plasma appears in the collision integral. This factor accounts for dynamical screening in the plasma, which ensures that collisions with impact parameter of order the Debye length $\lambda_D = \sqrt{\varepsilon_0 T / ne^2}$ or greater are exponentially damped. This effect demonstrates the well-known behavior of Debye screening [42], where the electric field from a point charge in a plasma will be exponentially damped on a length scale $\lambda_D$ by the rearrangement of the surrounding plasma. The Fokker-Planck collision operator diverges in the contribution from large-impact-parameter collisions, but by following Landau’s prescription from the original derivation [43] to cut the integration off at impact parameters $\lambda_D$ (which can be motivated by the Balescu-Lenard equation), one obtains a convergent integral. The contribution from small-angle collisions in the Fokker-Planck operator is then found to be larger than the contribution from large-angle collisions in the Boltzmann operator by a factor $\ln n \lambda_D^3 \simeq \ln \Lambda$, the so-called Coulomb logarithm.

In this chapter we will not pursue a detailed analysis of the BBGKY hierarchy of equations. Instead, we will derive the Boltzmann and Fokker-Planck collision operators with a heuristic argument, based on an analysis of binary collisions. This method gives the same result as the more rigorous derivation from first principles, and also provides some physical insight into how we may view collisions in a plasma.

### 2.2 The Boltzmann collision operator

The Boltzmann equation was originally derived by Ludwig Boltzmann in the late nineteenth century [44, 45] in order to study the dynamics of gases. As we indicated in the previous section, the Boltzmann equation for a plasma is valid when describing those large-angle collisions where the impact parameter is much smaller than the mean distance between particles in the plasma, that is for $\mu \ll 1$. The Boltzmann collision operator describes the rate-of-change of the distribution function due to
binary collisions, and we shall briefly derive it here in a form that will be suited to our applications.

We describe a binary interaction with a differential cross-section \(d\sigma_{ab}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}, \mathbf{p}')\) for particles \(a\) and \(b\) to be taken from initial momenta \(\mathbf{p}\) and \(\mathbf{p}'\), respectively, to final momenta \(\mathbf{p}_1\) and \(\mathbf{p}_2\), respectively. The cross-section is defined such that the total differential change of the phase-space particle density \(dn_a(t, \mathbf{x}, \mathbf{p}) = f_a(t, \mathbf{x}, \mathbf{p})d\mathbf{p}\) due to these interactions in a time interval \(dt\) is

\[
[dn_a(\mathbf{p})]_{c,ab} = f_a(\mathbf{p}_1)f_b(\mathbf{p}_2)g_\phi(\mathbf{p}_1, \mathbf{p}_2)d\sigma(\mathbf{p}, \mathbf{p}'; \mathbf{p}_1, \mathbf{p}_2)d\mathbf{p}_1d\mathbf{p}_2dt
- f_a(\mathbf{p})f_b(\mathbf{p}')g_\phi(\mathbf{p}, \mathbf{p}')d\sigma(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}, \mathbf{p}')d\mathbf{p}d\mathbf{p}'dt, \quad (2.4)
\]

where \(\mathbf{p}_1\) and \(\mathbf{p}_2\) are related to \(\mathbf{p}\) and \(\mathbf{p}'\) by the conservation of energy and momentum. The relativistic generalization of the relative speed \(v_{rel} = |\mathbf{v} - \mathbf{v}'|\), is the Møller relative speed \(g_\phi(\mathbf{p}, \mathbf{p}') = \sqrt{(\mathbf{v} - \mathbf{v}')^2 - (\mathbf{v} \times \mathbf{v}')^2/c^2}\) [46]. The collision operator can formally be defined as

\[
C_{ab}\{f_a, f_b\} \equiv \left(\frac{\partial^2 n_a}{\partial t \partial \mathbf{p}}\right)_{c,ab}
= \int d\mathbf{p}_1 f_a(\mathbf{p}_1) \int d\mathbf{p}_2 f_b(\mathbf{p}_2)g_\phi(\mathbf{p}_1, \mathbf{p}_2)\frac{\partial \sigma(\mathbf{p}, \mathbf{p}'; \mathbf{p}_1, \mathbf{p}_2)}{\partial \mathbf{p}}
- f_a(\mathbf{p}) \int d\mathbf{p}' f_b(\mathbf{p}')g_\phi(\mathbf{p}, \mathbf{p}')\sigma(\mathbf{p}, \mathbf{p}'), \quad (2.5)
\]

where the total cross-section \(\sigma(\mathbf{p}, \mathbf{p}')\) is defined as

\[\sigma(\mathbf{p}, \mathbf{p}') = \int d\mathbf{p}_1 \frac{\partial \sigma(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}, \mathbf{p}')}{\partial \mathbf{p}_1}.
\]

A symmetric form is obtained in the special case of elastic collisions by utilizing the principle of detailed balance [46], which is a symmetry relation for the cross-section stating that

\[g_\phi(\mathbf{p}_1, \mathbf{p}_2)d\sigma(\mathbf{p}, \mathbf{p}'; \mathbf{p}_1, \mathbf{p}_2)d\mathbf{p}_1d\mathbf{p}_2 = g_\phi(\mathbf{p}, \mathbf{p}')d\sigma(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}, \mathbf{p}')d\mathbf{p}d\mathbf{p}'.\]

Using this relation, which is valid for classical particles interacting with a central potential and also in the first order spin-averaged Born approximation in quantum mechanics [47, 48], Eq. (2.4) leads to

\[
C_{ab}\{f_a, f_b\} = \int d\mathbf{p}'d\sigma(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}, \mathbf{p}')g_\phi(\mathbf{p}, \mathbf{p}')\times
\left(f_a(\mathbf{p}_1)f_b(\mathbf{p}_2) - f_a(\mathbf{p})f_b(\mathbf{p}')\right). \quad (2.6)
\]
This is the operator which is typically known as the Boltzmann operator, although we shall apply the term more generally here to any integral operator of the form of Eq. (2.5). In Eqs. (2.4), (2.5) and (2.6) the first – the gain term – describes the rate at which particles \(a\) of initial momentum \(p_1\) are scattered into \(p\), while the second – the loss term – describes the rate at which particles scatter away from \(p\).

**Two-dimensional form** — The Boltzmann operator is inconvenient to work with because of the complicated relation between \((p, p')\) and \((p_1, p_2)\), combined with the singular nature of the Coulomb cross-section. However, it is significantly simplified under the conditions that
(i) the target particles are stationary, i.e. \(f_b(p) = n_b \delta(p)\);
(ii) the cross-section depends only on \(p_1, p\) and \(\cos \theta_s = p_1 \cdot p / p_1 p\), which excludes particles with internal degrees of freedom such as molecules, or spin-polarized plasmas;
(iii) and the distribution function is independent of gyroangle.
In this case, introducing again spherical coordinates \(p = (p, \cos \theta, \varphi)\) and \(p_1 = (p_1, \cos \theta_1, \varphi_1)\), the Boltzmann operator (2.5) takes the form
\[
C_{ab}\{f_a, f_b\}(t, p, \cos \theta) = n_b \int dp_1 \, v_1 f_a(t, p_1, \cos \theta_1) \frac{\partial \sigma}{\partial p}(p; p_1, \cos \theta_s) \nonumber
\]
\[
- n_b v f(t, p, \cos \theta) \sigma(p).
\]
The angle \(\cos \theta_s\) can be related to \(\cos \theta_1\), \(\cos \theta\), \(\varphi_1\) and \(\varphi\) by
\[
\cos \theta_s = \cos \theta_1 \cos \theta + \sin \theta_1 \sin \theta \cos(\varphi_1 - \varphi).
\]

As we will now show, the Boltzmann operator for stationary targets is diagonalized by the Legendre polynomials, making this a particularly convenient representation of the distribution function for the runaway problem. By introducing
\[
f_a(t, p, \cos \theta) = \sum_L f_L(t, p) P_L(\cos \theta),
\]
\[
C_{ab}(t, p, \cos \theta) = \sum_L C_L(t, p) P_L(\cos \theta),
\]
an application of the addition theorem for spherical harmonics,
\[
P_L(\cos \theta_s) = P_L(\cos \theta_1) P_L(\cos \theta)
\]
\[
+ \sum_{m=1}^L \frac{(L - m)!}{(L + m)!} \cos \left(m(\varphi_1 - \varphi)\right) P^m_L(\cos \theta_1) P^m_L(\cos \theta),
\]
2.2. THE BOLZMANN COLLISION OPERATOR

yields

\[ C_L(t,p) = n_b \int dp_1 p_1^2 v_1 f_L(p_1) 2\pi \int_{-1}^{1} d\cos \theta_s \, P_L(\cos \theta_s) \frac{\partial \sigma}{\partial p}(p; p_1, \cos \theta_s) \]

\[ - n_b v f_L(p) \sigma(p). \]  

(2.7)

This form for the Boltzmann collision operator is particularly suitable for numerical calculations, as the angular part is encoded in the Legendre modes in a simple way; the general form for a linear operator takes the form \( C(p, \cos \theta) = \sum_{L,L'} C_L \{ f_{L'} \} P_L(\cos \theta), \) but the Legendre polynomials diagonalize the collision operator in the sense that \( C_L \) depends only on the corresponding mode \( f_L \) of the distribution function. Numerically, this leads to a sparse matrix representation of the system, allowing a time and memory efficient treatment.

The Boltzmann operator given here is valid both for inelastic and elastic collisions, although in the elastic case \( \partial \sigma / \partial p \) will contain a delta function, as the kinematics then constrain \( p_1 = p_1(\cos \theta_s). \)

**Special case: Self-collisions** — Equation (2.7) is also valid for self-collisions with a simple modification to the last term (the sink term); in this case, the assumption of stationary targets can be viewed as a linearization around a cold bulk population, \( f_a = n_a \delta(p) + f_{a1}. \) This is a suitable description for scenarios where a relatively small population of energetic electrons is present, with momenta much greater than the thermal momentum. The collision operator then takes the form

\[ C_L(t,p) = n_a \int dp_1 p_1^2 v_1 f_L(p_1) 2\pi \int_{-1}^{1} d\cos \theta_s \, P_L(\cos \theta_s) \frac{\partial \sigma}{\partial p}(p; p_1, \cos \theta_s) \]

\[ - \frac{1}{2} n_a v f_L(p) \sigma(p) - \frac{1}{2} \frac{\delta(p)}{p^2} n_a \int dp' p'^2 v' \delta L,0 f_0(p') \sigma(p'). \]  

(2.8)

The new term at the end describes the removal of the initially stationary particle, and the factors 1/2 are introduced to avoid double counting collisions. From this expression the collision operator for knock-on collisions (avalanche generation) can be derived. For example, the model by Rosenbluth and Putvinski [8] follows by setting \( f_e(p) = n_{RE} \delta(\cos \theta - 1)\delta(p - p_0)/2\pi p_0^2 \) and letting \( p_0 \to \infty, \) i.e. by assuming that all runaways have infinite energy and zero pitch-angle.

**Special case: Heavy targets** — Another interesting special case is given by elastic collisions with infinitely heavy targets. In that case,
the energy of the light particle is conserved, and the cross-section takes the form \( \partial \sigma / \partial p = (\delta(p_1 - p) / 2\pi p^2) \partial \sigma / \partial \cos \theta_s \). This leads to a collision operator

\[
C_L(t, p) = -n_a v f_L(t, p) \int_{-1}^{1} d \cos \theta_s \left[ 1 - P_L(\cos \theta_s) \right] \frac{\partial \sigma(p, \cos \theta_s)}{\partial \cos \theta_s}.
\]

(2.9)

It is interesting to further reduce this in the case where the integral is dominated by the contribution from small scattering angles. We may then Taylor expand

\[
1 - P_L(\cos \theta_s) = \frac{L(L + 1)}{4} \theta_s^2 + O(\theta_s^4),
\]

which yields

\[
C_L(t, p) = -n_a v f_L(t, p) \frac{L(L + 1)}{4} \int_{0}^{\pi} d \theta_s \theta_s^3 \frac{\partial \sigma(p, \cos \theta_s)}{\partial \cos \theta_s}.
\]

Although the angles are assumed to be small, the integration is extended to \( \pi \) as the contribution from large angles is assumed to be negligible. We identify this as a familiar eigenvalue equation for the Legendre polynomials, yielding the collision operator

\[
C_{ab}(t, p) = -n_b v L\{f_a\} \int_{0}^{\pi} d \theta_s \frac{\theta_s^3}{2} \frac{\partial \sigma(p, \cos \theta_s)}{\partial \cos \theta_s},
\]

(2.10)

where the Lorentz operator \( L \) is half the pitch-angle part of the Laplace operator \( p^2 \nabla^2_p \):

\[
L\{f\} = \frac{1}{2} \frac{\partial}{\partial \xi} \left( (1 - \xi^2) \frac{\partial f}{\partial \xi} \right).
\]

This is an energy-conserving diffusion operator in pitch-angle. It is interesting to note that, in the Legendre-polynomial representation, the Boltzmann operator (2.9) and the diffusion operator (2.10) have the same structure – the only difference is that the coefficients are given by different integral moments of the scattering cross-section. Therefore the small-angle approximation in this limit does not provide a significant decrease in computational cost compared to the Boltzmann operator here, unlike the general case which we shall now treat.
2.3 The Fokker-Planck collision operator

When the interaction distance is significant compared to the mean particle separation, but the interaction is weak, the appropriate collision term is the Fokker-Planck operator, rather than the Boltzmann operator. However, as we will now show, the Fokker-Planck operator can in fact be derived from the Boltzmann operator in the limit of small momentum transfers in the collisions. That the seemingly opposite description of weak interactions in the Fokker-Planck picture can be contained in the Boltzmann picture of binary collisions appears counter-intuitive. It can, however, be physically understood by the fact that the small momentum transfers described by the Fokker-Planck operator only negligibly change the particle momentum in a single collision; then the net effect of the many-body interaction can be viewed as a linear superposition of pairwise momentum transfers [49].

The procedure is as follows: a general integral moment of the Boltzmann operator is

\[ J[\phi] = \int dp \phi(p) C_{ab} = \int dp \int dp' d\sigma(p_1, p_2; p, p') f_a(p)f_b(p') \times g_\phi(p, p')[\phi(p_1) - \phi(p)], \]

which is most easily seen by integrating Eq. (2.4) and switching names of the dummy variables \(p_1\) and \(p_2\) in the first term with \(p\) and \(p'\), respectively. For convenience we will suppress the arguments of \(d\sigma\) and \(g_\phi\) as they will remain unchanged for the rest of the calculation. Here we introduce the small-momentum-transfer argument: the integral is assumed to be dominated by the contribution from \(p_1 \approx p\). We then Taylor expand

\[ \phi(p_1) - \phi(p) \approx (p_1 - p) \cdot \frac{\partial \phi(p)}{\partial p} + \frac{(p_1 - p)(p_1 - p)}{2} : \frac{\partial \phi(p)}{\partial p \partial p}, \]

where we use dyadic notation such that the rank-2 tensor \(T = ab\) has components \(T_{ij} = a_i b_j\). By introducing the quantities

\[ \Delta p = p_1 - p, \]
\[ A = \int dp' g_\phi f_b(t, p') \int d\sigma \Delta p, \]  \(2.11\)
\[ D = \int dp' g_\phi f_b(t, p') \int d\sigma \Delta p \Delta p, \]  \(2.12\)
integrating by parts twice immediately yields

\[
J[\phi] = \int dp \phi(p) \left[ \frac{\partial}{\partial p} \cdot \left( -A(t, p)f_a(t, p) + \frac{1}{2} \frac{\partial}{\partial p} \cdot \left[ D(t, p)f_a(t, p) \right] \right) \right].
\]

As this equality holds for any \( \phi \), the small-momentum-transfer assumption therefore leads to the well-known Fokker-Planck operator \([50, 51]\)

\[
C_{ab}\{f_a, f_b\} = \frac{\partial}{\partial p} \cdot \left( -A_{ab}(t, p)f_a(t, p) + \frac{1}{2} \frac{\partial}{\partial p} \cdot \left[ D_{ab}(t, p)f_a(t, p) \right] \right).
\]

(2.13)

For collisions with infinitely heavy stationary targets, to leading order in scattering angle, this reduces to Eq. (2.10). For relativistic elastic electron-electron collisions, the Fokker-Planck operator was first given by Beliaev and Budker \([52]\), with a direct derivation from Eqs. (2.11), (2.12) and (2.13) later given by Akama \([53]\). The Fokker-Planck operator is conveniently expressed in the form

\[
C_{ab}\{f_e, f_e\} = \frac{\partial}{\partial p} \cdot \int dp' \mathcal{E} \cdot \left( \frac{\partial f_e(p)}{\partial p} f_e(p') - \frac{\partial f_e(p')}{\partial p'} f_e(p) \right),
\]

(2.14)

where the collision kernel \( \mathcal{E} \) is the symmetric rank-2 tensor \([54]\)

\[
\mathcal{E} = 2\pi \left( \frac{e^2}{4\pi \varepsilon_0} \right)^2 \ln \Lambda \frac{\gamma' \gamma (1 - \mathbf{v'}/c)^2}{c \left[ (\gamma'/\gamma - \mathbf{p'}/\mathbf{p}/(m_e c)^2] - 1 \right]^{3/2} \times \left\{ \left( \gamma' - \frac{\mathbf{p'}/\mathbf{p}}{m_e c^2} \right)^2 - 1 \right\}^{-1} - \frac{\mathbf{p} \mathbf{p} + \mathbf{p'} \mathbf{p'}}{m_e^2 c^2} + \left( \gamma' - \frac{\mathbf{p'}/\mathbf{p}}{m_e c^2} \right) \frac{\mathbf{p' \mathbf{p} + pp'}}{m_e^2 c^2} \right\},
\]

where \( I \) is the unit tensor and \( \gamma = \sqrt{1 + (p/m_e c)^2} \) is the relativistic Lorentz factor. In this expression, only the leading-order term in \( \ln \Lambda \) has been retained, which corresponds to the small-angle contribution to the integrals (2.11) and (2.12).

An approximate collision operator to study runaway electrons was developed in Ref. \([55]\). It is an asymptotic matching of the linearized Beliaev-Budker operator (2.14) in the high-energy limit with the non-relativistic collision operator \([43, 56]\) (corresponding to Eq. (2.14) for \( v \ll c \), linearized with a cold bulk of thermal velocity \( v_{Te} = \sqrt{2T_e/m_e} \ll c \)). The operator is

\[
C = \frac{B(p)}{p^2} L\{f_e\} + \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 \left( F(p)f_e + A(p) \frac{\partial f_e}{\partial p} \right) \right],
\]

(2.15)
where

\[ A(p) = \frac{m_e^2 c^2}{\tau_c} \frac{c}{v} G \left( \frac{v}{v_{Te}} \right), \]
\[ F(p) = 2 \frac{m_e c^2}{\tau_c} \frac{c^2}{v_{Te}^2} G \left( \frac{v}{v_{Te}} \right), \]
\[ B(p) = \frac{m_e^2 c^2}{\tau_c} \frac{c}{v} \left[ Z_{\text{eff}} + \phi \left( \frac{v}{v_{Te}} \right) - G \left( \frac{v}{v_{Te}} \right) + \frac{1}{2} \frac{v_{Te}^2}{c^2} v^2 \right]. \]

We have here introduced the collision time \( \tau_c^{-1} = n_e \ln \Lambda e^4/(4\pi\varepsilon_0^2 m_e^2 c^3) \), the error function \( \phi(x) = 2\pi^{-1/2} \int_0^x ds \exp(-s^2) \) and the Chandrasekhar function \( G(x) = (\phi(x) - x\phi'(x))/2x^2 \). A term proportional to the plasma effective charge \( Z_{\text{eff}} = \sum_i n_i Z_i^2/n_e \) (the sum taken over all ion species in the plasma) has been added to the pitch-angle scattering operator coefficient, which corresponds to the contribution from a set of stationary ion species, as in Eq. (2.10). For a non-relativistic bulk population, this collision operator has the correct asymptotic behaviour both as \( v \to 0 \) and \( \gamma \to \infty \), although the expression is never exact.

### 2.4 Synchrotron radiation reaction

In this section we will show how the effect of synchrotron radiation losses can be accounted for. In paper C, this has been used to find the energy which electrons can be accelerated to by an electric field before radiation losses stop their acceleration. The conditions for which a non-monotonic feature, a “bump”, can form in the tail of the runaway-electron distribution was also analyzed in the paper. The presence of such a feature can destabilize the plasma, and also effectively limits the maximum electron energy. This makes bump formation an interesting attribute of the solutions to the kinetic equation to study.

In the kinetic equation (2.2), it has been assumed that the only force acting on a particle is the Lorentz force \( \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \) due to the presence of an electric field \( \mathbf{E} \) or a magnetic field \( \mathbf{B} \). However, this description can be generalized to account for a general force \( \mathbf{F} = \mathbf{F}(\mathbf{p}) \) by replacing [57]

\[ q(\mathbf{E} + \mathbf{v} \times \mathbf{E}) \cdot \frac{\partial f}{\partial \mathbf{p}} \mapsto \frac{\partial}{\partial \mathbf{p}} \cdot \left[ \mathbf{F}(\mathbf{p}) f(\mathbf{p}) \right]. \quad (2.16) \]

With this replacement, Eq. (2.1) still takes the form of a continuity equation in phase space, and reduces to the familiar equation when the force \( \mathbf{F} \) is chosen as the divergence-free Lorentz force.
The energy which runaway electrons lose by emitting radiation limits the maximum energy that they can reach [58], which shows that this effect must be carefully accounted for in order to understand the dynamics of runaway electrons. In the late nineteenth century, around the same time as Boltzmann derived his famous kinetic equation for dilute gases, it was discovered that – according to Maxwell’s equations – a charged particle in accelerated motion will emit radiation [59]. Synchrotron radiation is the radiation emitted by a charged particle moving near the speed of light in a circular motion [60, 61]. In magnetized plasmas, synchrotron radiation is therefore emitted by runaway electrons due to their gyrating motion around the magnetic field. The synchrotron radiation reaction force can be understood classically by accounting for the electromagnetic radiation emitted by a particle in accelerated motion, which leads to the Abraham-Lorentz-Dirac (ALD) force [62, 63, 64]

\[
F_{ALD} = \frac{q^2\gamma^2}{6\pi\varepsilon_0c^3} \left[ \ddot{v} + \frac{3\gamma^2}{c^2} (v \cdot \dot{v})\ddot{v} + \frac{\gamma^2}{c^2} \left( v \cdot \dot{v} + \frac{3\gamma^2}{c^2} (v \cdot \dot{v}) \right) v \right].
\] (2.17)

This formula was simplified in Ref. [65] for the case of magnetized plasmas, where the motion is dominated by gyromotion such that \(v \cdot \dot{v} \approx 0\), and using the Landau method [61] of neglecting the acceleration by the ALD-force itself. This approximation yields the force components (in a spherical coordinate system, when averaged over the gyromotion)

\[
\begin{align*}
\left( \frac{dp}{dt} \right)_{ALD} &= -(1 - \xi^2) \frac{\gamma p}{\tau_r}, \\
\left( \frac{d\xi}{dt} \right)_{ALD} &= (1 - \xi^2) \frac{1}{\gamma\tau_r}, \\
\frac{1}{\tau_r} &= \frac{e^4 B^2}{6\pi\varepsilon_0 m_e^3 c^5}.
\end{align*}
\]

Thus, a non-isotropic reaction force is produced by the emission of synchrotron radiation. At high energy, the retarding component is approximately \(dp/dt \approx p^2_\perp/m_e c \tau_r\), and increases rapidly in magnitude with perpendicular momentum. However, as the electric field only accelerates runaways in the parallel direction, force-balance alone cannot explain how synchrotron radiation losses limit the maximum runaway energy. It is the pitch-angle scattering due to collisions that increases the perpendicular momentum of the electrons, which together with the subsequent synchrotron emission causes all electrons to eventually reach a steady-state velocity distribution. This steady state often exhibits a “bump” in
the tail of the distribution, beyond which the distribution decays exponentially with momentum.

2.5 CODE

In order to study the momentum-space dynamics of runaway electrons, a numerical tool CODE (COllisional Distributions of Electrons [66]) has recently been developed. The code obtains solutions to the kinetic equation (2.3) with the collision operator (2.15), treated as an initial-value problem. By representing the distribution function in terms of Legendre polynomials in pitch-angle cosine and a finite-difference discretization of the momentum coordinate, a flexible and computationally efficient scheme is obtained. The model contains only the essential physics required in order to study a wide range of momentum-space runaway dynamics, making it highly suited for studies such as those presented in this thesis.

Various models for large-angle collisions are implemented in CODE, which can all be derived from the Boltzmann equation (2.8). As previously mentioned, the model by Rosenbluth and Putvinski [8] follows by assuming the runaway distribution function to be of the form

\[ f_e(t, p, \xi, \varphi) = \frac{n_{RE}(t)}{2\pi p^2} \delta(p - p_0)\delta(\cos \theta - 1), \tag{2.18} \]

and then taking the limit \( p_0 \to \infty \) of the resulting collision operator. That is, all runaways are assumed to be infinitely energetic, and have no perpendicular momentum. The basic assumption here is that the runaways will have energies much larger than the thermal energy, and also larger than the knock-on particles – which predominantly have energies corresponding to the critical runaway speed (see Fig. 1.1). This model can be powerful for example in the later phase of a runaway discharge after a tokamak disruption, where most of the runaways have had time to be accelerated to highly relativistic (\( \gamma \gg 1 \)) energies, but is not suited for studying the initial runaway phase where most electrons have near-thermal energies.

A more powerful model for knock-on collisions was derived by Chiu et al. [67] by relaxing some assumptions of the preceding work, instead using the form

\[ f_e(t, p, \xi, \varphi) = \frac{1}{2\pi} f_e(t, p) \delta(\cos \theta - 1) \tag{2.19} \]
for the distribution function. While still neglecting the perpendicular momentum of the runaways, it accounts fully for the energy distribution which makes it applicable both for low energies in the initial runaway phase, as well as for large runaway energies where it reduces to the Rosenbluth-Putvinski model.

These have been compared in detail in paper B, using kinetic simulations with CODE to determine the differences between the models. Agreeing qualitatively with an approximate analytic model, it is shown that the difference can indeed be large when the runaway momentum does not far exceed the thermal momentum; sometimes the Chiu model yields higher avalanche runaway rates, and sometimes lower, depending on the strength of the electric field.

In paper B we also analyze the runaway growth by hot-tail generation, which occurs during a rapid temperature drop – electrons which were initially in the far tail of the hot thermal population may become runaway due to their lower collisionality, allowing them to retain their energy during the temperature drop. In the paper, previous non-relativistic studies of hot-tail generation [68] were extended to the relativistic equation which CODE solves, and the effect of the electric field on the hot-tail generation was also investigated. Differences in runaway growth of up to almost an order of magnitude were demonstrated in the more complete treatment achievable using CODE, compared to previous findings.

We have now briefly described the basic kinetic theory needed for electron-runaway investigations, introducing primarily well-known concepts and results. Based on this framework, we will in the following chapter develop a powerful method of accounting for the effect of bremsstrahlung emission.
Chapter 3

Bremsstrahlung

When charged particles collide, the resulting emission is referred to as bremsstrahlung (German for “braking radiation”, as it causes the particles to decelerate) [69, 70]. In this chapter we will describe in detail how the effect of bremsstrahlung emission on the motion of runaway electrons can be accounted for in plasmas. This question has recently been investigated in the context of magnetic-fusion plasmas in Refs. [58, 71], where an approximate model for the bremsstrahlung losses was used in order to determine the maximum energy reached by runaway electrons. In paper A we have extended their work by introducing a more realistic framework for bremsstrahlung losses based on the Boltzmann operator, which produces a qualitatively different response of the electrons compared to the previous studies.

Unlike synchrotron radiation – where typical frequencies are low enough that the energies of individual photons can be ignored – it is found that at relativistic electron energies the frequency of the emitted bremsstrahlung radiation corresponds to photon energies comparable to the electron energy. Because of this, to describe the bremsstrahlung emission from highly relativistic runaways, a quantum-mechanical description is necessary. The quantum-mechanical treatment was first described in detail in an extensive 1934 paper by Bethe and Heitler [72]. In the quantum-mechanical picture, bremsstrahlung is the result of a binary interaction between two charged particles resulting in the emission (creation) of one or more photons. The analysis of Bethe and Heitler provides a differential cross-section for the process $\mathbf{p}_1 \rightarrow \mathbf{p}, \mathbf{k}$, where $\mathbf{p}_1$ and $\mathbf{p}$ are the incident and outgoing electron momenta, respectively, and $\mathbf{k}$ is the photon momentum. The target is treated as a stationary
scattering center, meaning that we neglect the recoil of the target which would generally cause modifications of order $\gamma m_e/M$ when $M$ is the target mass.

This description of bremsstrahlung as a binary collision process allows us to describe its effect on the electron distribution with the Boltzmann collision operator. The differential cross-section $\partial\sigma/\partial p$ (in the form that it appears in the collision integral (2.5)) for bremsstrahlung interactions in the Born approximation was originally published by Racah [73], with a crucial misprint that was corrected by McCormick et al. [74] 22 years later. The interactions, of the form $p_1 \mapsto p + k$, satisfy conservation of energy $\sqrt{1 + p_1^2/m_e^2c^2} = \sqrt{1 + p^2/m_e^2c^2} + kc$, however, momentum is not conserved, as any amount can be transferred to the infinitely heavy scattering center (the nucleus). The cross-section formula is given by

$$\frac{\partial\sigma_{ab}}{\partial p} = Z_0^2 \alpha r_0^2 \frac{2k}{p_1 \gamma} W(p; p_1, \cos \theta_s)$$

$$W(p; p_1, \cos \theta_s) = \frac{2\gamma_1 \gamma + (\gamma_1^2 + \gamma^2 - 1)\lambda - \lambda^2}{k^2 \lambda^2 \sqrt{\lambda(\lambda + 2)}} \ln \left(1 + \lambda + \sqrt{\lambda(\lambda + 2)}\right)$$

$$- \frac{2\gamma_1 \gamma - \lambda}{k^2 \lambda^2} - \frac{3(\gamma_1^2 \gamma^2 - 1)^2}{\lambda^2 p_1^4 p^4}$$

$$+ \frac{4(\gamma_1^2 \gamma^2 - \gamma_1 \gamma + 1) - \gamma_1 \gamma (p_1^2 + p^2) + (\gamma_1^2 + \gamma^2 + \gamma_1 \gamma - 1)\lambda}{2\lambda^2 p_1^2 p^2}$$

$$+ \left(\frac{2 \gamma_1 \gamma - 1}{\lambda^3} - \frac{k^2}{\lambda^4}\right) \frac{2(\gamma_1^2 + \gamma^2 - \gamma_1 \gamma)p_1^2 p^2 + 3k^2(\gamma_1 + \gamma)^2}{p_1^4 p^4}$$

$$+ \frac{l}{p^3} \left[\frac{\gamma + 2\gamma^3}{\lambda^2 p^2} + \frac{2\gamma^4 + 2p_1^2 p^2 + \gamma_1 (\gamma_1 + \gamma) - (\gamma_1 \gamma + p^2)\lambda}{2k \lambda^2}\right]$$

$$+ \gamma \left(\frac{2 \gamma_1 \gamma - 1}{\lambda^3} - \frac{k^2}{\lambda^4}\right) \frac{2\gamma_1 p^2 - 3k \gamma^2}{kp^2}\right]$$

$$+ \frac{l_1}{p_1^3} \left[\frac{\gamma_1 + 2\gamma^3}{\lambda^2 p_1} - \frac{2\gamma_1^4 + 2p_1^2 p_2 + \gamma_1 (\gamma_1 + \gamma) - (\gamma_1 \gamma + p_1^2)\lambda}{2k \lambda^2}\right]$$

$$- \gamma_1 \left(\frac{2 \gamma_1 \gamma - 1}{\lambda^3} - \frac{k^2}{\lambda^4}\right) \frac{2\gamma_1 p_1^2 + 3k \gamma_1^2}{kp_1^2}\right].$$

(3.1)

In the expression for $W$, the electron momenta $p_1$ and $p$, and photon momentum $k$, have been normalized to $m_e c$ for clarity. The fine-structure constant is denoted $\alpha = e^2/(4\pi\varepsilon_0 hc) \approx 1/137$, and $r_0 =$
$e^2/(4\pi\varepsilon_0 m_e c^2) \approx 2.8 \cdot 10^{-15}$ m is the classical electron radius. We have also introduced the auxiliary quantities

\[
\begin{align*}
l &= \ln(\gamma + p), \\
l_1 &= \ln(\gamma_1 + p_1), \\
\lambda &= \gamma_1 \gamma - p_1 p \cos \theta_s - 1,
\end{align*}
\]

where the full angular dependence of the cross-section is captured in $\lambda$. This is the simplest bremsstrahlung formula that provides a complete and self-consistent description. The validity of the Born approximation is limited to $v/c \gtrsim Z\alpha$ and $v_1/c \gtrsim Z\alpha$, i.e. both the incident and outgoing electron must be sufficiently fast, otherwise the plane-wave assumption in the Born approximation will be violated. For runaways typically moving near the speed of light, in plasmas where $Z \ll 100$, this is well satisfied for the incident velocity. The condition on the outgoing velocity, however, puts an upper limit on photon energies for which the formula is valid. The correction when $v/c$ is comparable to $\alpha Z$ can approximately be accounted for by multiplying the cross-section formula with the so-called Elwert factor [75]

\[
F_E = \frac{v}{v_1} \frac{1 - \exp(-2\pi\alpha Z c/v_1)}{1 - \exp(-2\pi\alpha Z c/v)}.
\]

A thorough analysis of bremsstrahlung emission is given in Ref. [76] (and references therein), on which our current discussion is primarily based. More sophisticated models of bremsstrahlung can be obtained by numerical methods, of which a few examples are: a full partial-wave expansion solution of the Dirac equation; Elwert-Haug theory where the lowest-order wavefunction is taken as Coulomb-problem free states instead of plane waves; or accounting for screening in the Born approximation by including the effect of bound electrons through an atomic form factor. The latter can be studied analytically in the limit of complete screening [72], but an analytic expression for $\partial\sigma/\partial p$ has, to our knowledge, not been published.

In addition, the formula given in Eq. (3.1) is only strictly valid for electron-ion bremsstrahlung; for electron-electron interactions, exchange and retardation effects are important, and complicate the analysis greatly. The full quadruply differential cross-section was originally given in a Ph.D. thesis [77], but a reprint can also be found in Ref. [76]. Despite a lengthy formula covering more than 5 full pages, it
has been analytically integrated over electron and photon emission angles in Refs. [78, 79]. The expression for $\partial \sigma / \partial p$ needed for the collision operator does, however, not appear to exist in the published literature. Because of this, we also apply Eq. (3.1) to electron-electron bremsstrahlung. Beyond the prohibitive complexity of the full formula, this can be further motivated by the fact that in the high-energy limit, the full electron-electron and electron-ion bremsstrahlung formulas produce the same total radiation cross-section, indicating that our choice is suitable for a first approximation.

### 3.1 Screening

In many scenarios of interest the ions in the plasma will not be fully ionized. For example, during tokamak disruptions the temperature may drop to a few electronvolts, which is lower than typical atomic ionization energies, and large quantities of high-atomic-number gases are sometimes injected as a disruption mitigation strategy [17, 19]. The presence of electrons which remain bound to the nuclei in a plasma have a so-called screening effect, as the electron cloud effectively cancels (partially or totally, depending on the degree of ionization) the charge of the nucleus, as seen from afar. The importance of screening can be estimated in the following way. The differential cross-section in the Born approximation involves the Fourier transform $\tilde{V}$ of the scattering potential $V$,

$$\tilde{V}(q) = \int \text{d}r V(r) e^{iq \cdot r / \hbar}.$$  

Here, $q = p_1 - p - k$ is the momentum transferred to the nucleus, and by the conservation of energy $p = \sqrt{(\gamma_1 - k)^2 - 1} = \sqrt{p_1^2 - 2\gamma_1 k + k^2}$ (again using normalized units). The effect of screening will be important when significant contributions to $\tilde{V}$ originate from distances of the order of the atomic size or greater, i.e. near the Bohr radius $r \sim a_0 = \hbar / m_e c \alpha$ (with $\alpha$ again the fine-structure constant). For sufficiently large $q$, the exponential factor in the integrand will be rapidly oscillating and the contribution will vanish; therefore, the minimum value of $q$ sets the length-scale that is probed in the interaction. For $q_{\text{min}} \lesssim \hbar / a_0 = \alpha m_e c$ screening effects are important, as the bound electrons of the atom are then probed. The minimum momentum transfer (corresponding to a maximum probed radius) is given when the vectors $p_1$, $p$ and $k$ are
aligned, and has the value

\[ q_{\text{min}} = p_1 - p - k \approx \frac{k}{2\gamma_1 \gamma}, \quad (3.2) \]

where in the last step large energies were assumed, \( p_1 \gg m_e c \). Note the seemingly counter-intuitive behavior that the momentum transfer to the ion decreases with increasing electron energy, a direct consequence of the conservation laws involving the creation of a third particle. The threshold for the importance of screening effects is then given by the condition

\[ \frac{k}{2\gamma_1 (\gamma_1 - k)} \lesssim \alpha. \]

Approximately 80% of the contribution to the total radiative stopping power (see Eq. (3.3)) comes from photon energies greater than 10% of \( \gamma_1 \). Therefore, setting \( k = 0.1 \gamma_1 \), we find the condition \( \gamma_1 \gtrsim 7.6 \) for screening effects to be important. This means that Eq. (3.1) must be modified for highly relativistic electron energies in the presence of ions which are not fully ionized. The modification will act to reduce the rate of bremsstrahlung interactions compared to when the full nuclear charge is inserted into the Bethe-Heitler formula (3.1). Here we will, however, restrict the study to fully ionized plasmas or sufficiently low electron energies where these effects can be neglected. For more details on how atomic screening effects collisions, a thorough analysis of screening in elastic collisions has recently been given in Ref. [80].

### 3.2 Low-energy photon contribution

Bremsstrahlung reactions resulting in the emission of photons with energies comparable to the incident-electron energy are responsible for the dominant contribution to the radiative stopping power, which is proportional to the energy moment of the cross-section [72];

\[ \langle k \sigma \rangle = \int dk \, k \frac{\partial \sigma}{\partial k}. \quad (3.3) \]

Indeed, for an incident electron of energy \( \gamma_1 \), the relative contribution to the integral from those photons with energy \( k \leq k_0 \ll \gamma_1 \) is of order \( k_0/\gamma_1 \). Therefore it is commonly argued that the low-energy photons contribute negligibly to the fast-electron dynamics [81]. However, the
above argument only proves that the energy loss is small; the reactions are kinematically allowed to change the direction of the incident electron arbitrarily, as the nucleus acts as a momentum sink. As the cross-section for small \( k \) goes as \( d\sigma \propto 1/k \), it is clear that many reactions involving low-energy photons occur, indicating that this may be a significant effect.

The singularity in the cross-section also proves to be a challenge when numerically evaluating the Boltzmann operator, as the gain and loss terms are both infinitely large. However, with the method described below a simplified model can be derived, where the singularity is analytically resolved by considering the contribution from the low-energy photons separately, and utilizing the smallness of \( k/p_1 \), the ratio of photon to electron momentum.

To leading order in \( k/p_1 \), for large energies \( \gamma_1 \gg 1 \), the bremsstrahlung cross-section (3.1) takes the form

\[
\frac{\partial \sigma}{\partial \cos \theta_s \partial k} = \frac{1}{k} W(\gamma_1, \gamma),
\]

where \( W(\gamma_1, \gamma) = W(\gamma, \gamma_1) = W(\gamma_1) \) is symmetric, and \( k = \gamma_1 - \gamma \). We can rewrite the Boltzmann operator, Eq. (2.7), by writing

\[
\frac{\partial \sigma}{\partial p} = \frac{1}{2\pi p \gamma} \frac{\partial \sigma}{\partial k \partial \cos \theta_s},
\]

and changing variables in the integral, \( dp_1 = (\partial p_1/\partial k)dk = dk/v_1 \). This yields the equivalent form

\[
C_L(t, p) = n_b \int dk \cos \theta_s \frac{p_1^2}{p \gamma} f_L(p_1) P_L(\cos \theta_s) \frac{\partial \sigma}{\partial k \partial \cos \theta_s}
- n_b \int dk \cos \theta_s v f_L(p) \frac{\partial \sigma}{\partial k \partial \cos \theta_s},
\]

where the symmetry property of the cross-section was (indirectly) used in the bottom line. Now, to leading order in \( k/p_1 \) we also have \( p_1 = p \), which finally yields the operator

\[
C_L(t, p) = -n_b v f_L(p) \int d \cos \theta_s \left[ 1 - P_L(\cos \theta_s) \right] \frac{\partial \sigma}{\partial \cos \theta_s}, \tag{3.4}
\]

where

\[
\frac{\partial \sigma}{\partial \cos \theta_s} = \int dk \frac{\partial \sigma}{\partial k \partial \cos \theta_s} = 2\pi \gamma p \int d\sigma(p; p_1, \cos \theta_s) \frac{\partial \sigma(p; p_1, \cos \theta_s)}{\partial p}.
\]
The operator thus obtained is the stationary-target elastic-scattering operator of Eq. (2.9), which is expected as we have neglected the energy carried by the photons. For small angles $\theta \lesssim 1/\gamma_1$, the cross-section $\partial \sigma / \partial \cos \theta_s$ goes as $1/\theta^2$. This is in contrast with the cross-section for purely elastic Coulomb collisions, which goes as $1/\theta^4$, indicating that the low-energy photon contribution to bremsstrahlung will not be so dominated by small-angle collisions. Indeed, a significant contribution to the integrals in $C_L$ are given by scattering angles $\theta_s \sim 1$, showing that a Fokker-Planck treatment is inadequate. The vanishing of the $L = 0$ term (corresponding to the spherically symmetric part of the operator, since $P_0(x) = 1$) ensures that these reactions contribute only to pitch-angle deflection, and not energy loss, consistent with the argument at the beginning of the section and the result obtained in Ref. [81].

To quantify the importance of the low-energy photon effect, we analytically evaluate the $L = 1$ term of Eq. (3.4) and compare it to the corresponding term of the elastic Coulomb-scattering operator. The latter is proportional to the term containing $Z_{\text{eff}}$ in Eq. (2.15). In the limit $\gamma_1 \gg 1$ and $k/\gamma_1 \ll 1$, after a tedious integration of the full cross-section formula in Eq. (3.1), the ratio between the bremsstrahlung and Coulomb expressions is found to be

$$\frac{C_{\text{small-k}}^1}{C_{\text{elastic}}^1} = \alpha \frac{2 \ln \Lambda_B}{\pi \ln \Lambda} \left[ \left( \ln \frac{2p}{m_e c} - 1 \right)^2 + 1 \right].$$

(3.5)

We have introduced the bremsstrahlung logarithm $\ln \Lambda_B$, which arises in a manner analogous to the Coulomb logarithm for elastic collisions and is due to the logarithmic divergence with $k$ of the bremsstrahlung cross-section. If the operator is constructed to account for all bremsstrahlung emissions of energy $k \leq k_0$, the bremsstrahlung logarithm is defined as $\ln \Lambda_B = \ln(k_0/k_c)$, where $k_c$ is a lower cut-off in photon energy which has a physical origin, to be discussed below. The ratio in Eq. (3.5) decreases monotonically with $L$ (which can be verified numerically by performing the integration in Eq. (3.4)), indicating that for the comparison it is sufficient to evaluate the $L = 1$ term.

### 3.3 Lower limit in photon energy

We will now discuss the integration limits in the evaluation of $\partial \sigma / \partial \cos \theta_s$. The operator (3.4) will be used to account for those bremsstrahlung emissions with photon energy $k \leq k_0$, where $k_0$ is some arbitrary cutoff
satisfying \( k_0 \ll \gamma_1 - 1 \). However, the integral is singular, and needs to be cut off at some lower limit as well, which we denote \( k_c \). There are three mechanisms which need to be considered in determining this lower limit:

1. Polarization of the background medium – in a semi-classical treatment, scattering in a dielectric medium can be modeled by multiplying the cross-section by a suppression factor depending on the dielectric constant of the medium. This will have the effect of effectively screening those interactions in which the radiation is emitted with frequency below the plasma frequency, thus effectively cutting off the integral.

2. Many-photon emissions – at low photon energies, many-photon emissions will become increasingly important for the reaction rate. It is known that this quantum-electrodynamical effect naturally resolves the singularity in the cross-section.

3. Stimulated emission and absorption – at sufficiently low photon energy, there will be a high enough number of photons that they will significantly interact with the plasma before escaping. This has the effect of limiting the validity of our equation to photon energies above some critical value.

We shall discuss these effects in turn, and investigate what the respective cutoff energies are.

**Polarization of the background medium:** The effect on the cross-section of the polarization of the background medium was originally treated in Ref. [82]; it was experimentally verified by the SLAC-E-136 accelerator [83], and the effect (together with related phenomena) has been discussed at length in a more recent review article [84]. The argument goes as follows: as we discussed in connection with the screening effect of bound electrons, there is a length-scale associated with a bremsstrahlung reaction, related to the momentum \( q \) transferred to the ion by \( l = \hbar / q \) (which is the wavelength of the virtual photon exchanged between electron and nucleus). Here, however, we are interested in the parallel length scale \( l_B = \hbar / q_\parallel \), which is sometimes referred to as the *formation length*. The parallel momentum transfer is

\[
q_\parallel = \frac{p_1}{p_1} \cdot (p_1 - p - k) = p_1 - p \cos \theta_s - k \cos \theta_1,
\]
where $\cos \theta_s = p_1 \cdot p_1/p_1 p$ and $\cos \theta = p_1 \cdot k/p_1 k$. As we are now concerned about the contribution from photons carrying momentum much smaller than the electron momenta, we Taylor expand this formula to first order in $k/p_1$, using $p = \sqrt{(\gamma_1 - k)^2 - 1}$. This gives

$$q_\parallel = p_1 (1 - \cos \theta_s) + k \left( \frac{1}{v_1} \cos \theta_s - \cos \theta_1 \right).$$

If we here expand in large electron energy $\gamma_1 \gg 1$ and small angles $\theta_s \sim \theta_1 \ll 1$, we find

$$q_\parallel = \frac{k}{2 \gamma_1^2} + \gamma_1^2 \left( \frac{\theta_1^2}{2} - \theta_s^2 \right).$$

As expected, for vanishing scattering angles (when electron and photon momenta are aligned), we obtain again the minimum momentum transfer of Eq. (3.2). If we neglect the contribution due to the angular deviation, i.e. for $(\gamma_1 \theta)^2 \ll k/\gamma_1$, we find a formation length

$$l_B = \frac{2 \hbar \gamma_1^2}{k}.$$

The analysis in Ref. [84] shows that the effect is covered by the introduction of a suppression factor $S$, defined so that

$$\frac{\partial \sigma}{\partial p} = S \left( \frac{\partial \sigma}{\partial p} \right)_0,$$

where $(\partial \sigma/\partial p)_0$ is the cross-section of Eq. (3.1) with polarization effects unaccounted for. The suppression factor $S$ is given by the ratio between formation lengths when the photon energy $k$ is replaced by $\sqrt{\epsilon k}$ (to account for the change in the speed of light of photons in the medium) and when it is left unchanged. That is, we evaluate

$$S = \frac{p_1 - \cos \theta_s \sqrt{\gamma_1 - k} - 1 - k \cos \theta_1}{p_1 - \cos \theta_s \sqrt{\gamma_1 - k} - 1 - \sqrt{\epsilon k} \cos \theta_1},$$

$$\epsilon(k) = 1 - \frac{\hbar^2 \omega_p^2}{\omega^2},$$

where the high-frequency limit of the dielectric tensor $\epsilon$ is used. In the limit where angular deflection is ignored, we find a suppression factor

$$S(k) = \frac{1}{1 + (\gamma \hbar \omega_p/kc)^2},$$
indicating that the cross-section is effectively cut off at photon momenta $k = \gamma \hbar \omega_p/c$, or when the frequency of the radiation is $\gamma \omega_p$. However, as mentioned, this is valid only when $(\gamma_1 \theta)^2 \ll k/\gamma_1$. In practice, the low photon-energy operator acquires significant contributions from $k/\gamma_1$ much smaller than 1%, and when $\gamma_1 \theta$ is of order unity, or significantly larger. Therefore we are in fact often in the opposite limit, where the momentum transfer to the nucleus depends only on the scattering angle, and not on the photon energy. Hence the above argument (with an effective cut-off in parallel momentum transfer) will sometimes not cause a cut-off in photon energy, as the suppression factor stays close to unity.

However, if we anyway apply this model which has been commonly used in the literature and was experimentally observed, and since the final expression is only logarithmically sensitive to our choice, it is found that the effect of background polarization ensures that radiation with frequency below the plasma frequency $\omega_p = \sqrt{n_e e^2/m_e \varepsilon_0}$ will be suppressed. This corresponds to photon energies $k_e c = \hbar \omega_p$, or in normalized units

$$\frac{k_e}{m_e c} = \frac{\hbar \omega_p}{m_e c^2} = 7.3 \cdot 10^{-10} \sqrt{n_{20}},$$

where $n_{20} = n_e/(10^{20} \text{ m}^{-3})$. Introducing $\kappa_0 = k_0/m_e c$ as a normalized photon energy, the bremsstrahlung logarithm is given by

$$\ln \Lambda_B = \ln \frac{\kappa_0 m_e c^2}{\hbar \omega_p} = \ln \frac{\sqrt{m_e \varepsilon_0 \kappa_0 m_e c^2}}{\sqrt{n_e e^2 \hbar}} \approx 21.0 - \ln \frac{\sqrt{n_{20}}}{\kappa_0}. \tag{3.6}$$

For typical runaway scenarios in fusion plasmas, where $\gamma$ ranges from 10 to 100 and $n_{20}$ ranges from 0.1 to 10, and if we choose $\kappa_0 = \gamma/1000$, the bremsstrahlung logarithm $\ln \Lambda_B$ takes values in the range 15 to 20.

2. Many-photon emissions: We have discussed the bremsstrahlung cross-section in lowest-order perturbation theory, which corresponds to only accounting for bremsstrahlung processes where a single photon is emitted. This is known to break down (in the infamous infrared divergence of QED [85]) as the photon energies approach zero, where multi-photon processes contribute significantly. However, these multi-photon processes can be elegantly accounted for with a simple modification to the cross-section. To show this, we present an argument from Ref. [86]: assume that the cross-section is determined by a measurement device that can measure only photons with energy above some minimum energy $E_l$. Then, the observed cross-section is obtained by integrating
over all multi-photon processes where photons of energy less than \( E_l \) are involved. This procedure yields the correction

\[
\frac{\partial \sigma}{\partial p} = F(q) \left( \frac{\partial \sigma}{\partial p} \right)_0,
\]

(3.7)

where \( q^2 = (p - p_1)^2 - (\gamma - \gamma_1)^2 \), and

\[
F(q) = \exp \left[ -\frac{\alpha}{\pi} f_{\text{IR}}(q^2) \ln \frac{q^2}{E_l^2} \right],
\]

\[
f_{\text{IR}}(q^2) = \int_0^1 dx \frac{1 + q^2/2}{1 + q^2 x(1 - x)} - 1
= \frac{2 + q^2}{q \sqrt{q^2 + 4}} \ln \frac{\sqrt{q^2 + 4 + q}}{\sqrt{q^2 + 4 - q}} - 1.
\]

The effect of soft-photon processes is then negligible whenever

\[
\frac{\alpha}{\pi} f_{\text{IR}}(q^2) \ln \left( \frac{q^2}{E_l^2} \right) \ll 1. \tag{3.8}
\]

The left-hand side is monotonically increasing with \( q \), indicating that we need to verify that the inequality (3.8) is well satisfied for the case where \( q \) is the greatest. In our case of low-energy photon emissions, \( q \sim |p - p_1| \) is the momentum transferred to the nucleus, which is maximized by large-angle collisions where \( q \sim p \). For \( q \gg 1 \), the asymptotic form of the infrared correction is \( f_{\text{IR}} = \log q^2 \). We then obtain the condition

\[
\frac{\alpha}{\pi} \ln \frac{p^2}{m_e c^2} \ln \frac{p^2 c^2}{E_l^2} \ll 1.
\]

Solving for \( p \) then yields

\[
\frac{p}{m_e c} \ll \sqrt{\frac{E_l}{m_e c^2}} \exp \sqrt{\left( \log \sqrt{\frac{E_l}{m_e c^2}} \right)^2 + \frac{\pi}{4\alpha}}.
\]

As we will consider “hard photons” (primary ones) emitted with frequencies as low as the plasma frequency, we will be interested in determining what the effect on the cross-section is of soft-photon emissions with lower energy. If we can demonstrate that the cross-section is negligibly affected in this worst-case scenario, we can safely ignore the effect
3.3. LOWER LIMIT IN PHOTON ENERGY

entirely. Therefore, setting \( E_l = \hbar \omega_p \) we obtain the condition

\[
\frac{p}{m_e c} \ll 2.7 \cdot 10^{-5} \sqrt{n_{20}} \exp \left( \frac{10.5 - \frac{1}{4} \ln n_{20}}{2} \right) + 108
\]

\[
\simeq 70.1(n_{20})^{1/5.6} \quad (\text{for } \ln n_{20} \ll 40).
\]

Because of this, for typical runaway energies of order 10-100 MeV, we see that the soft-photon contribution will become important approximately at the plasma-frequency scale of photon energies, but only for the largest-angle reactions (where \( q \sim p \)). Note that the inequality is in fact logarithmic, indicating that this effect will also affect the process at lower energies. However, note that this is a pessimistic estimate. In fact, a large contribution to the collision operator is given by interactions with emission angles \( \theta_s \sim 1/\gamma \), for which the momentum transfer \( q \) is of order \( m_e c \). At this scale, the soft-photon correction will always be negligible in practical scenarios.

3. Stimulated emission and absorption: It is well known that existing photons in the plasma can interact with the electrons by the related processes of stimulated emission and absorption. These effects are proportional to the number of photons in the plasma, and will therefore be increasingly significant for low-energy photons, as the cross-section then grows as \( 1/k \). To ensure the validity of our proposed bremsstrahlung operator, we need to verify that these effects are small in the scenarios that we consider.

In Ref. [87] the electron bremsstrahlung Boltzmann operator is given, accounting for spontaneous and stimulated emission and absorption, as

\[
C d\mathbf{p} = n_i \int d\mathbf{p}' d\nu d\Omega_p f_e(\mathbf{p}') \tilde{\alpha}_\nu(\mathbf{p}'; \mathbf{n}, \mathbf{e}) \left[ 1 + \frac{c^2}{2h\nu^3} I_\nu(\mathbf{n}) \right] \\
+ n_i \int d\mathbf{p}' d\nu d\Omega_p f_e(\mathbf{p}') \tilde{\beta}_\nu(\mathbf{n}, \mathbf{p}'; \mathbf{e}) I_\nu(\mathbf{n}) \\
- n_i \int d\mathbf{p} d\nu d\Omega'_p f_e(\mathbf{p}) \tilde{\alpha}_\nu(\mathbf{p}; \mathbf{n}, \mathbf{e}') \left[ 1 + \frac{c^2}{2h\nu^3} I_\nu(\mathbf{n}) \right] \\
- n_i \int d\mathbf{p} d\nu d\Omega'_p f_e(\mathbf{p}) \tilde{\beta}_\nu(\mathbf{n}, \mathbf{p}; \mathbf{e}') I_\nu(\mathbf{n}).
\]

We here follow the original notation in [87], where the essential part is the specific intensity of radiation \( I_\nu \), which is related to the photon
distribution function $\phi(r, k, t)$ by

$$\phi(r, k, t) = \frac{c^2}{h^4\nu^3} I_\nu(n, r, t).$$

The emission and absorption coefficients are denoted $\tilde{\alpha}$ and $\tilde{\beta}$, respectively, and are related by the three-body version of the principle of detailed balance [48, 87]

$$\tilde{\beta}_\nu(n, p; e') = \frac{v'}{v} \frac{c^2}{2h\nu^3} \tilde{\alpha}_\nu(p'; n, e).$$

In this formula, the terms involving $\tilde{\alpha}$ but not $I_\nu$ correspond to spontaneous emission, as described by our operator, while the rest account for absorption and stimulated emission. Therefore, the relative importance of these effects compared to the spontaneous emission that we have considered is of order

$$\frac{\text{effect of absorption and stimulated emission}}{\text{effect of spontaneous emission}} \sim \frac{c^2}{2h\nu^3} I_\nu = \frac{h^3 \phi}{2}.$$

Thus we see that the effect is expected to be small when the mean phase-space volume occupied by photons is less than two Planck units, i.e. when

$$\frac{h^3 \phi}{2} \ll 1.$$

To complete the argument we only need to find an estimate for the photon distribution function, $\phi$. We can do so by using the following order-of-magnitude estimate:

The rate at which photons are created, per unit volume and momentum, is given by

$$\frac{dn_\phi(t, r)}{dt d\mathbf{k}} = \sum_b n_b(t, r) \int d\mathbf{p} f_e(t, r, \mathbf{p}) v \frac{\partial \sigma_{eb}}{\partial \mathbf{k}},$$

where $\partial \sigma_{eb}/\partial \mathbf{k}$ is the bremsstrahlung differential cross-section (with respect to emitted-photon energy and direction), which can be found for example in Ref. [88], and the sum is taken over all particle species in the plasma. Since the cross-section goes as $1/k$ for small $k$, it will be sufficient to show that our assumption is justified in the low-photon-energy
3.3. LOWER LIMIT IN PHOTON ENERGY

limit where the photon distribution is expected to be the largest. The differential cross-section is

$$\frac{\partial \sigma}{\partial k} \sim \frac{1}{4\pi k^2} \frac{\partial \sigma}{\partial k} \sim \frac{4}{3\pi} \frac{\alpha Z^2 r_0^2}{k^3} \ln \frac{2p^2}{m_e c k},$$

as obtained from Eq. (3.1) integrated over electron angles (a calculation initially performed in Ref. [73]) in the low-\(k\), high-\(p\) limit. Assuming a mono-energetic relativistic electron beam with momentum \(p\), and the plasma to be confined in a volume \(V = L^3\), the total number of photons created per second per momentum-space volume \(k\) is (summed over particle species)

$$\frac{dN_\phi}{dt d\mathbf{k}} \approx \frac{4}{3} V n_e n_{RE} (1 + Z_{\text{eff}}) r_0^2 \frac{1}{137\pi} \ln \frac{2p^2}{m_e c k}.$$ 

Multiplying the expression by \(L/c\) yields the total number of photons in the volume, which divided by \(V\) gives the distribution function:

$$\phi(r, k, t) \sim \frac{4}{3} L n_e n_{RE} (1 + Z_{\text{eff}}) r_0^2 \frac{1}{137\pi} \ln \frac{2p^2}{m_e c k}. \quad (3.9)$$

At the minimum value of \(k = k_c = \hbar \omega_p/c\), we thus require the smallness of

$$\frac{h^3}{2} \phi \sim \frac{16\pi^2}{3} \frac{1 + Z_{\text{eff}}}{137} \frac{L r_0^2 n_e n_{RE} c^3}{\omega_p^3} \left(2 \ln \frac{p}{m_e c} - \ln \frac{\hbar \omega_p}{2m_e c^2}\right)$$

$$\approx 4.6(1 + Z_{\text{eff}}) \frac{L n_{RE, 20}}{\sqrt{n_{20}}} \left(21.7 + 2 \ln \frac{p}{m_e c}\right), \quad (3.10)$$

where \(L\) is in meters and densities are in units of \(10^{20}\) m\(^{-3}\). As an upper limit for reasonable values of this quantity in laboratory plasmas, we take a post-disruption runaway scenario with \(Z_{\text{eff}} = 15\), \(L = 10\) m, \(n_{RE} = 10^{-3} n_e\) and \(n_e = 50 \cdot 10^{20}\) m\(^{-3}\), with \(p = 200 m_e c\). Eq. (3.10) then takes the value 170, which means that there are many photons per Planck unit of phase-space volume, and absorption and stimulated emission can thus be expected to have a significant effect on those photons with the lowest energy. Note, however, that due to the \(1/k^3\) sensitivity to photon energy in Eq. (3.9), less than an order of magnitude above \(\hbar \omega_p\) the photon distribution will again be negligible. If we were to instead cut the integration off at the photon energy where \(h^3 \phi\) is small, \(\ln \Lambda_B\) would typically change by less than 10%, because of the weak logarithmic sensitivity to \(k_c\) in the bremsstrahlung logarithm.
In a more realistic scenario, one could have $L \sim 1 \text{ m}$, $n_e \sim 10^{20} \text{ m}^{-3}$, $n_{\text{RE}} \sim 10^{-4} n_e$ and $p \sim 50 m_e c$, giving $h^3 \phi / 2 \sim 0.2$. Then absorption and stimulated-emission effects become important close to the plasma frequency, making $\omega_p$ again a suitable cut-off point when these effects are accounted for.

In conclusion, it appears that in laboratory plasmas the lower limit $k_c$ in photon energy is well described by considering only the screening effect of scattering in a dielectric medium, where the plasma frequency effectively cuts off the bremsstrahlung reaction rate. The QED effect of (soft) many-photon emissions typically becomes important at higher electron energies, or significantly lower plasma densities, while stimulated emission and absorption processes become important for larger systems, or higher runaway densities. Therefore the bremsstrahlung differential cross-section can be cut off near the plasma frequency, which introduces the bremsstrahlung logarithm $\ln \Lambda_B$ from Eq. (3.6) which enhances the contribution from photons which carry energies much smaller than the electron energy.
Chapter 4

Ion runaway

In this chapter we will describe ion runaway, which is a process related to electron runaway. This will provide an introduction to the theory underlying the work presented in paper D, which numerically treats the ion kinetic equation to determine the conditions for ion runaway. For this purpose, the tool CODE for electron-runaway studies was extended to solve the ion kinetic equation, resulting in the new open-source code CODION (COllisional Distributions of IONs [89, 90]).

Ion runaway is a phenomenon similar to electron runaway in many ways. The initial runaway-generation mechanism is the same – a sufficiently strong electric field can overcome the collisional friction force of the thermal bulk and accelerate a subpopulation of ions to high energy. However, there are a few key differences between ion runaway and electron runaway:

1. The ions are not the lightest particle species in the plasma; the ion-electron collisions are qualitatively different to the electron-ion collisions.

2. The collisional friction force on a runaway ion is strongly non-monotonic, with the consequence that, unless the electric field is comparable to the Dreicer field, the kinetic energy of runaway ions is bounded from above because of electron friction.

3. A non-relativistic treatment is sufficient, and radiation losses are negligible.

The kinetic description of runaway ions can be constructed in the following way. The particle species in a homogeneous plasma each satisfy
a kinetic equation.

\[ \frac{\partial f_a}{\partial t} + \frac{q_a}{m_a} E \frac{\partial f_a}{\partial v_||} = \sum_b C_{ab}\{f_a, f_b\}, \]  

(4.1)

where the sum on the right-hand side is taken over all particle species \( b \) in the plasma. We will consider only that ion species which is accelerated at the highest rate, in order to be able to assume that the other ion species remain near equilibrium (i.e. that they are well described by a Maxwellian distribution function). However, this cannot be assumed for self-collisions of this species, nor for ion-electron collisions. Instead the self-collision operator can be linearized by writing \( C_{aa}\{f_a, f_a\} \approx C_{aa}\{f_a, f_{Ma}\} + C_{aa}\{f_{Ma}, f_a\} \) where \( f_{Ma} \) is the Maxwellian distribution for particle species \( a \) with density \( n_a \) and temperature \( T_a \), we obtain the kinetic equation

\[ \frac{\partial f_a}{\partial t} + \frac{q_a}{m_a} E \frac{\partial f_a}{\partial v_||} = C_{aa}\{f_{Ma}, f_a\} + C_{ae}\{f_a, f_e - f_{Me}\} 
+ \sum_b C_{ab}\{f_a, f_{Mb}\}. \]

As argued in Ref. [57], the perturbed electron distribution \( f_e - f_{Me} \) varies over velocities much greater than the ion velocity, and therefore yields a collision operator that describes a uniform friction force. For an ion species \( i \), the ion-electron collision operator is

\[ C_{ie}\{f_i, f_e - f_{Me}\} = -\frac{R_{ie}}{m_i n_i} \cdot \frac{\partial f_i}{\partial v}, \]

where \( R_{ie} = \int dv \, m_i v C_{ie} \) is the mean ion-electron friction force. This assumes that \( f_{Me} \) is chosen as a Maxwellian in the rest frame of the ions, otherwise the operator \( C_{ae}\{f_a, f_{Me}\} \) will contribute additional friction that would need to be accounted for in the above term. Due to the conservation of momentum in collisions, the ion-electron friction force is related to the electron-ion friction by \( R_{ie} = -R_{ei} \), and \( R_{ei} \) can readily be calculated from the electron kinetic equation.

Taking the momentum moment \( \int dv \, m_e v \ldots \) of the electron kinetic equation yields the force-balance equation

\[ \frac{\partial (n_e m_e V_e)}{\partial t} = -n_e e E + \sum_j R_{ej}, \]  

(4.2)
where the sum is taken over all ion species \( j \) (the electron-electron contribution vanishing due to momentum conservation in collisions). Due to the large mass ratio \( m_j/m_e \), the ions are generally much slower than the electrons. As a consequence, the dependence on ion species in the electron-ion friction is to leading order entirely captured in the collision frequency \( \nu_{ej} \propto n_j Z_j^2 \), yielding

\[
\sum_j R_{ej} = R_{ei} \sum_j \frac{n_j Z_j^2}{n_i Z_i^2} = \frac{n_e Z_{\text{eff}}}{n_i Z_i^2} R_{ei}.
\]

The electrons equilibrate with the electric field on a time scale much shorter than that of the ion runaway process, due to their lower mass. Therefore, unless significant electron-runaway generation occurs which effectively acts as a momentum sink, the time-derivative can be neglected in the force-balance equation (4.2), yielding

\[
R_{ie} = -R_{ei} = -\frac{Z_i}{Z_{\text{eff}}} n_i q_i E.
\]

In the ion kinetic equation this readily allows the ion-electron collision operator to be combined with the electric-field term. Equation (4.1) can then be written

\[
\frac{\partial f_i}{\partial t} + \frac{q_i}{m_i} E^* \frac{\partial f_i}{\partial v_\parallel} = C_{ii}\{f_{Mi}, f_i\} + \sum_b C_{ib}\{f_i, f_{Mb}\},
\]

\[
E^* = \left(1 - \frac{Z_i}{Z_{\text{eff}}} \right) E_\parallel,
\]

and describes a population being accelerated by the effective electric field \( E^* \), and only experiencing collisions with Maxwellian background species (and the field-particle contribution from self-collisions in order to ensure the conservation of momentum and energy in such collisions). The effective electric field thus arises due to friction between ions and the electrons drifting in the electric field.

In a pure plasma, \( Z_i = Z_{\text{eff}} \), and complete cancellation occurs between electric field and electron friction; no ion runaway is possible in this case. Only in impure plasmas does a finite effective electric field remain, and indeed, for an ion species with \( Z_i \gg Z_{\text{eff}} \) the effective electric field can exceed the original electric field in magnitude, and is anti-parallel to the electric field. In this scenario, runaway occurs due to acceleration by electron friction rather than by the electric field.
As we consider the initial-value problem starting from a thermal equilibrium, we will typically work in the initial rest frame. As the runaway-ion population builds up, the total friction against the Maxwellian electrons may end up being comparable to the friction \(-Z_i/Z_{\text{eff}}Z_i eE\) against the drifting electrons, at which point the model is no longer valid. This, together with the use of a linearized self-collision operator, puts an upper boundary on the ion runaway densities we may consider.

### 4.1 Ion friction-force estimates

Valuable physical insight into the ion runaway process can be obtained by considering the friction force acting on a test-ion moving through the plasma. Formally, the test-particle equations of motion can be obtained by considering velocity moments of the kinetic equation for a delta distribution \(f_a = \delta(v - u(t))\) representing the test particle [91]. This method was pursued in Refs. [24, 26] where ion runaway in solar flares was considered, and later expanded upon in paper D to consider general plasma compositions. The momentum moment of the ion kinetic equation yields the test-particle equation of motion

\[
\frac{\partial (m_i v)}{\partial t} = Z_i eE^* \xi - \frac{m_i v T_i}{\tau_{ie}} \left( \frac{Z_{\text{eff}} + \bar{n}}{v^2/v_T^2} + \frac{4}{3\sqrt{\pi}} \sqrt{\frac{m_e T_e^3}{m_i T_i^3 v T_i}} \right),
\]

where we use the pitch-angle cosine \(\xi = v_\parallel / v\), the ion-electron collision frequency \(\tau_{ie}^{-1} = n_e \ln \Lambda Z_i^2 e^4/(4\pi Z_i m_i v T_i^2)\), and introduced the quantity \(\bar{n} = \sum_j n_j Z_j^2 m_j/(n_e m_i)\). Here explicit expressions for the collision operator with a Maxwellian background species [91] have been used, under the assumption that the velocities satisfy \(v_{Tj} \ll v \ll v_T\) for all ion species \(j\). The term containing the parentheses represents collisional friction, in which the first term expresses ion-ion friction which decreases with velocity and dominates for low velocities, whereas the second term describing ion-electron friction increases with velocity and dominates at high velocities.

The solutions \(\partial (m_i v)/\partial t = 0\) represent those velocities where electric-field acceleration exactly balances collisional friction. For electric fields

\[
E^* > E_{\min}^* = 2 \frac{m_i v T_i}{Z_i e\tau_{ie} T_e} \left( \frac{3}{2\pi} \frac{m_e}{m_i} (Z_{\text{eff}} + \bar{n}) \right)^{1/3},
\]
two solutions \( v_{c1} \) and \( v_{c2} \) exist which describe the runaway velocity above which an ion is accelerated by the electric field, and the maximum velocity before electron friction overcomes the electric field, respectively. Therefore, ions with velocity \( v_{c1} < v < v_{c2} \) will be accelerated, before accumulating at \( v_{c2} \).

However, it should be noted that the above test-particle equation of motion is not unique. By considering the energy moment \( \int d\mathbf{v} (m_i v^2 / 2) \ldots \) of the ion kinetic equation, one instead obtains

\[
\frac{\partial (m_i v)}{\partial t} = Z_i e E^\ast \xi - \frac{m_i v T_i}{\tau_{ie}} \left[ \frac{v_{Ti}^2}{v^2} + \frac{4}{3\sqrt{\pi}} \sqrt{\frac{m_e T_i^3}{m_i T_e^3}} \left( \frac{v}{v_{T_i}} - \frac{3T_e}{T_i} \frac{v_{T_i}}{v} \right) \right].
\]

If we assume that \( (v/v_{T_i})^2 \gg 3T_e/T_i \), this reduces to the momentum-moment equation (4.2) with the simple exchange \( \bar{n} \mapsto Z_{\text{eff}} + \bar{n} \). This equation may provide more accurate estimates of the critical velocities; using the same procedure to estimate the electron runaway velocity yields the well-known formula \( v_{c_e}/v_{Te} = \sqrt{E_D/2E} \), while the momentum-balance equation incorrectly gives a result which is larger by a factor \( \sqrt{2} \). The discrepancy may be understood by the fact that pitch-angle scattering contributes to friction in the momentum-balance equation, but not in the energy-balance equation as it is an energy-conserving effect. The angular deflection will not stop a particle from running away (except sometimes indirectly), and hence the energy-balance equation provides a better estimate. The substitution \( (Z_{\text{eff}} + \bar{n}) \mapsto \bar{n} \) may thus improve the results given in Refs. [24, 26] and paper D, although these estimates should perhaps primarily be viewed as a guide to interpret solutions of the kinetic equation, and to make qualitative predictions regarding the features of the ion runaway distribution.

Note finally the limits to the validity of the model described here. The linearization of the self-collision operator requires small runaway densities, corresponding to short times or electric fields \( E^\ast \sim E_{\text{min}}^\ast \). Extending far above \( E_{\text{min}}^\ast \) requires the use of a non-linear self-collision operator. At the same time, the electric field must be sufficiently weak to avoid significant runaway-electron generation which would affect \( E^\ast \), therefore requiring \( E \lesssim 0.1E_D \).
Chapter 5

Conclusions and outlook

Conclusions

Runaway is an important phenomenon, which occurs in both terrestrial and space plasmas. It is of particular interest in magnetic-fusion research where runaway electrons can strike the wall after being accelerated to highly relativistic energies, at which point they can cause severe damage to the plasma-facing components of the device. Runaway is also of interest in space and astrophysical applications, where they may be responsible for observed gamma-ray emissions.

In this thesis, we have described a kinetic model of runaway in plasmas, accounting for acceleration by an electric field, elastic Coulomb collisions and dissipation by radiation. We have discussed how a combination of a Boltzmann collision operator and Fokker-Planck operator is needed in order to describe both the large-angle collisions which lead to runaway avalanches, as well as the small-angle collisions that otherwise dominate the collisional dynamics. The framework described here is applicable to model the runaway dynamics of both electrons and ions.

An existing numerical tool which solves the electron kinetic equation in a homogeneous plasma, CODE (COollisional Distribution of Electrons) [66], has been extended to include a range of new effects. The radiation reaction force due to synchrotron emission in a magnetized plasma was described in chapter 2 within the framework of the electromagnetic Abraham-Lorentz-Dirac force, and its effect on the dynamics of runaway electrons was considered in paper C. It is shown that the synchrotron reaction force has a tendency to form a non-monotonic “bump” in the runaway distribution, which effectively also limits the maximum
energy of the electrons. A condition on plasma parameters for when such bumps can form – and at what energy – was derived, which is of importance in magnetic-fusion research as such features have the potential to destabilize the plasma.

In paper B the runaway dynamics was considered in plasmas where temperatures are rapidly varying in time. This causes hot-tail runaway generation to occur, and by generalizing previous studies to also account for relativistic corrections and the effect of an electric field, it was demonstrated that CODE is well-suited to study this phenomenon. In the same paper, a comparison between different knock-on collision operators found in the literature was also performed. It was demonstrated that, while the more accurate model of Chiu et al. [67] reduces to the simpler Rosenbluth-Putvinski model [8] when the runaway energies are large, at lower energies the avalanche growth rates differ significantly. At low electric fields ($E \sim E_c$) the former model yielded runaway growth rates as small as an order of magnitude lower than the simpler model, while at high electric field ($E \gg E_c$) more than twice the runaway rate was observed.

In chapter 3 the effect of bremsstrahlung emission was treated as a binary collision using the Boltzmann collision operator, which is necessary since the emitted photons often have energies comparable to the electron energy. The bremsstrahlung collision operator is explored in careful detail, as bremsstrahlung losses have previously only been accounted for in plasmas using approximate methods, where either photon energies or electron directions have been neglected. Accounting for both of these effects causes qualitatively new behavior of the electrons, such as enhanced pitch-angle scattering caused by the emission of low-energy photons.

In paper A the effect of bremsstrahlung is investigated with a numerical study of the kinetic equation, using CODE in which the bremsstrahlung collision operator has been implemented. It is revealed that – unlike in previous studies where all runaways accumulate at a well-defined energy where energy loss from bremsstrahlung equals the gain from the electric field – broad runaway-electron energy distributions are obtained. A significant fraction of electrons are consistently found to be more than twice as energetic as predicted in the previous approximate model.

A sister code to CODE has been developed which solves the kinetic equation for runaway ions, using similar numerical methods. The new
tool, CODION (COllisional Distribution of IONs) [90], includes the physics described in chapter 4 and is utilized in paper D to study runaway acceleration of ions, and under what conditions this occurs. The ion-runaway growth rate is investigated for various plasma compositions and electric fields for a solar-flare like scenario. It is also demonstrated that significant ion acceleration by the runaway mechanism alone is unlikely to occur during tokamak disruptions due to the large electric fields and long acceleration times required.

Papers A-D utilize a linearized equation, which limits the investigations to those scenarios where the electrons are sufficiently close to equilibrium. In order to study scenarios with large electric fields, or with rapidly time-varying plasma parameters, non-linear methods are required. In paper E, a novel method of numerically solving the non-linear Landau-Fokker-Planck equation is presented. By expressing the distribution function approximately as a finite sum of (non-orthogonal) Gaussian basis functions \( f_n(p) = c_n \exp \left[ -\frac{(v - v_n)^2}{2mT_n} \right] \), the non-linear Fokker-Planck equation can be represented as a quadratic algebraic equation where the coefficients take the form of simple, analytically known functions. The conservation properties of density, momentum and energy of the numerical scheme is investigated, as well as the relaxation of an initial multi-peaked distribution into the Maxwellian equilibrium distribution.

Outlook

The Boltzmann transport equation has rarely been considered in the context of runaway modelling with continuum kinetic models (unlike Monte Carlo particle-following codes, where the corresponding processes are more naturally accounted for). We have presented a procedure to simplify the equation in the case of a cylindrically symmetric plasma near equilibrium, enabling accurate modelling of collisional processes with large momentum transfers in a computationally efficient way.

In the future this framework can be applied to investigate other runaway phenomena, which are not yet fully understood. For example, the models which have previously been used to consider knock-on collisions suffer from certain defects:

- The knock-on model by Rosenbluth and Putvinski [8] shows unphysical behavior, such as the creation of an infinite amount of momentum and energy per second in the plasma.
• The improved model by Chiu et al. [67], which was also used in paper B, resolves this issue but still breaks momentum and energy conservation by only following the target particle – the effect of the knock-on collisions on the incoming fast electron is ignored.

• The pitch-angle distribution of the runaway distribution is ignored in both models, and it is instead assumed that all electrons move purely in the parallel direction.

The Boltzmann operator described here is the necessary tool to develop a self-consistent and complete model for avalanche generation which suffers from none of these problems.

The impact on runaway-electron dynamics of partially ionized atoms in the plasma is another phenomenon which can be investigated using this framework. The screening effect of the bound electrons is analogous to the description in Section 3.1, which indicates that for momentum transfers greater than \( q \sim \hbar/a_0 \) the bound electrons may be ignored, and a runaway electron will feel the full charge of the nucleus. In a collision this implies that in such plasmas, large-angle electron-ion collisions will be enhanced relative to the small-angle collisions, hence a Boltzmann approach such as Eq. (2.9) is required in future studies in order to describe this effect accurately.

We thus see that there are several phenomena of importance to runaway where the methods developed in this work can be used to further improve current understanding.
Bibliography


[90] CODION is an open-source tool, available for download on GitHub. http://github.com/Embreus/CODION

Paper A

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Effect of bremsstrahlung radiation emission on fast electrons in plasmas,
Submitted for publication in New Journal of Physics.
Effect of bremsstrahlung radiation emission on fast electrons in plasmas

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Bremsstrahlung radiation emission is an important energy loss mechanism for energetic electrons in plasmas. In this paper we investigate the effect of spontaneous bremsstrahlung emission on the momentum-space structure of the electron distribution, fully accounting for the emission of finite-energy photons. We find that electrons accelerated by electric fields can reach significantly higher energies than what is expected from energy-loss considerations. Furthermore, we show that the emission of soft photons can contribute significantly to the dynamics of electrons with an anisotropic distribution.

Energetic electrons are ubiquitous in plasmas, and bremsstrahlung radiation is one of their most important energy loss mechanisms [1, 2]. At sufficiently high electron energy, around a few hundred megaelectronvolts in hydrogen plasmas, the energy loss associated with the emission of bremsstrahlung radiation dominates the energy loss by collisions. Bremsstrahlung emission can also strongly affect electrons at lower energies, particularly in plasmas containing highly charged ion species.

An important electron acceleration process, producing energetic electrons in both space and laboratory plasmas, is the runaway mechanism [3]. In the presence of an electric field which exceeds the minimum to overcome collisional friction [4], a fraction of the charged particles can detach from the bulk population and be accelerated to high energies, where radiative losses become important. Previous studies of laboratory plasmas [5, 6] and lightning discharges [7] have shown that the energy carried away by bremsstrahlung radiation is important in limiting the energy of runaway electrons. The effect of bremsstrahlung radiation loss on energetic-electron transport has also been considered in astrophysical plasmas, for example in the context of solar flares [8]. However, only the average bremsstrahlung friction force on test particles has been considered in these studies. In this paper, we present the first quantitative kinetic study of how bremsstrahlung emission affects the runaway-electron distribution function.

Starting from the Boltzmann electron transport equation, we derive a collision operator representing bremsstrahlung radiation reaction, fully accounting for the finite energy and emission angle of the emitted photons. We implement the operator in a continuum kinetic-equation solver [9], and use it to study the effect of bremsstrahlung on the distribution of electrons in 2D momentum-space. We find significant differences in the distribution function when bremsstrahlung losses are modeled with a Boltzmann equation (referred to as the “Boltzmann” or “full” bremsstrahlung model), compared to the model where only the average friction force is accounted for (the “mean-force” model). In the former model, the maximum energy reached by the energetic electrons is significantly higher than is predicted by the latter. In previous treatments which considered average energy loss [5–7] or isotropic plasmas [2], the emission of soft (low-energy) photons did not influence the electron motion. We show that in the general case, emission of soft photons contributes significantly to angular deflection of the electron trajectories.

We will treat bremsstrahlung as a binary interaction (“collision”) between two charged particles, resulting in the emission of a photon [1]. We shall describe the effect of such collisions on the rate of change of the distribution function $f_a(t, \mathbf{x}, \mathbf{p})$ of some particle species $a$ at time $t$, position $\mathbf{x}$ and momentum $\mathbf{p}$, defined such that $n_a(t, \mathbf{x}) = \int d\mathbf{p} f_a(t, \mathbf{x}, \mathbf{p})$ is the number density of species $a$ at $\mathbf{x}$. In what follows we suppress the time- and space dependence of all functions, as the collisions will be assumed local in space-time, and we shall consider only spatially homogeneous plasmas.

The collision operator $C_{ab}^B(f_a, f_b)$ describing the rate of change of the distribution function due to bremsstrahlung interactions between species $a$ and $b$ is given by $C_{ab}^B = (\partial f_a/\partial t)_{c,ab} = \int (dn_a)_{c,ab} / dt d\mathbf{p}$, where the differential change $(dn_a)_{c,ab}$ in the phase-space density due to collisions in a time interval $dt$ is given by [10, 11]

$$
(dn_a)_{c,ab} = f_a(p_1) f_b(p_2) g_0 d\sigma_{ab} d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}' / dt - f_a(p) f_b(p') g_0 d\sigma_{ab} d\mathbf{p} d\mathbf{p}' / dt.
$$

(1)

Here, $d\sigma_{ab} = d\sigma_{ab}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{k}, \mathbf{p}, \mathbf{p}')$ is the differential cross-section for a particle $a$ of momentum $\mathbf{p}$ and a particle $b$ of momentum $\mathbf{p}'$ to be taken to momentum $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, while emitting a photon of momentum $k/c$. We have also introduced the Møller relative speed $g_0 = \sqrt{(\mathbf{v} - \mathbf{v}')^2 - (\mathbf{v} \times \mathbf{v}')^2 / c^2}$. The barred quantities $\bar{d}\sigma$ and $\bar{g}_0$ are defined likewise, but with $(\mathbf{p}, \mathbf{p}')$ and $(\mathbf{p}_1, \mathbf{p}_2)$ exchanged. Eq. (1) accounts only for the effect on the distribution of the spontaneous emission of photons; interactions with existing photons by absorption and stimulated bremsstrahlung emission will be neglected here. The correction to the collision operator by these processes is described in [12]: the effect is negligible when $\phi(\mathbf{x}, \mathbf{p}) \ll 2 / h^3$, where $h$ is Planck’s constant and $\phi$ is the distribution function of photons. An estimate of the photon distribution function shows that
the corrections are important for sufficiently dense, or large, plasmas; however, for the special case of electron runaway during tokamak disruptions, which is of particular concern, the corrections may be safely neglected. In other scenarios it is primarily bremsstrahlung processes involving low-energy photons that may be affected. The collision operator then takes the form

\[ C_{ab}^{\text{B}}(p) = \int dp_1 f_a(p_1) \int dp_2 g_b(p_2) \frac{\partial \sigma_{ab}}{\partial p} \]  

where \( \sigma_{ab} = \int dp_1 (\partial \sigma_{ab}/\partial p_1) \) is the total cross-section. A significant simplification to (2) occurs if (i) target particles can be assumed stationary, \( f_a(p) = n_0 \delta(p) \); and (ii) the plasma is cylindrically symmetric (and spin unpolarized). \( f_a(p) = f_s(p, \cos \theta) \), where \( \cos \theta = p_\parallel / p \) and \( p_\parallel \) is the Cartesian component of \( p \) along the symmetry axis. Then the differential cross-section \( \partial \sigma_{ab}/\partial p \) for an electron to scatter from momentum \( p \) into \( p_1 \) with the emission of a photon, depends only on \( p, p_1 \) and \( \cos \theta = p_\parallel / p \). The resulting operator can be conveniently expressed in terms of an expansion in Legendre polynomials \( P_L \). We write \( f_s(p) = \sum_L f_L(p) P_L(\cos \theta) \) and \( C_{ab}^{\text{B}}(p) = \sum_L C_L^{\text{B}}(p) P_L(\cos \theta) \), and obtain

\[ C_L^{\text{B}}(p) = n_b \int dp_1 \left[ \frac{2}{\pi} L(1) \right] \frac{1}{2} \int_0^1 d \cos \theta_s \]

\[ \times P_L(\cos \theta_s) \frac{\partial \sigma_{ab}}{\partial p} - n_b v f_L(p) \sigma_{ab}(p). \]  

The integration limits in \( p_1 \) are determined by the conservation of energy, giving \( m_e c^2 \sqrt{(\gamma + k/m_e c^2)^2 - 1} < p_1 < \infty \). In this work we use the differential cross-section \( \partial \sigma / \partial p \) for scattering in a static Coulomb field in the Born approximation, integrated over photon emission angles. This expression was first derived by Racah [13], with a misprint later corrected in [14]. For the Boltzmann model we use this full cross-section is employed, while for the mean-force model we use the high-energy limit as in [5–7].

A useful approximation to the collision operator is obtained by neglecting the deflection of the electron in the bremsstrahlung reactions, formally achieved by the replacement \( \partial \sigma / \partial p = [\delta(\cos \theta_s - 1) - 2 \pi \sigma] / \partial p \), where \( \delta(\cos \theta_s - 1) = \sum_p 2 \pi \sigma / (\partial \sigma / \partial p) \), which yields

\[ C_L^{\text{B}}(p) \approx n_b \int dp_1 v f_s(p_1, \cos \theta) \frac{\partial \sigma}{\partial p}(p, p_1) \]

\[ - n_b v f_s(p, \cos \theta) \sigma(p). \]  

The bremsstrahlung cross-section has an infrared divergence; for low photon energies \( k \), it diverges logarithmically as \( d \sigma \propto 1/k \). The total energy loss rate is however finite, indicating that a large number of photons carrying negligible net energy are emitted. A consequence of this behavior is that the two terms in the Boltzmann operator (2) are individually infinitely large, necessitating the introduction of a photon cut-off energy \( k_0 \), below which the bremsstrahlung interactions are ignored in (3) and (4). We can however proceed analytically to evaluate the effect of the low-energy photons. While they carry little energy, they may contribute to angular deflection, analogously to the small-angle collisions associated with elastic scattering. Taylor expanding (3) in small photon energy \( k = \gamma_1 - \gamma \) yields to leading order

\[ C_L^{\text{small-}k} = -n_b v f_L(p) \int_{k_0}^{k_0} dk \int_1^1 d \cos \theta_s \]

\[ \times \left[ 1 - P_L(\cos \theta_s) \right] \frac{\partial \sigma}{\partial p} d \sigma / d \theta / d \cos \theta_s. \]  

Since \( P_0(\cos \theta_s) \equiv 1 \), the angle-averaged electron distribution (represented by the \( L = 0 \) term) is not directly affected by the low-energy photons, reflecting the fact that the photons carry negligible energy, consistent with the description by Blumenthal & Gould [2] for the isotropic case. Due to the logarithmic divergence of the cross-section, however, a significant contribution to angular deflection (represented by the \( L \neq 0 \) terms) is possible. Inspection of the integrand in (5) further reveals that significant contributions originate from large-angle scatterings, indicating that a Fokker-Planck approximation is inappropriate. While it may seem counter-intuitive that low-energy photon emissions contribute to large-angle collisions, note that due to the large mass ratio between electron and ion, large momentum transfers to the nucleus is allowed even without energy transfer. When the electron energy exceeds the ion rest energy, however, ion recoil effects will modify (5).

We can quantify the importance of the low-energy photons by calculating the \( L = 1 \) term of (5) – giving the loss rate of parallel momentum – and comparing it to the corresponding term of the elastic-scattering collision operator given in [9]. Carrying out the integration, one obtains the ratio

\[ \frac{C_L^{\text{small}}}{C_L^{\text{elastic}}} = \frac{2}{\pi} \frac{\Delta R_s}{\ln \Lambda} \left( \frac{2 \pi \sigma}{m_e c^2} \right)^2 \frac{\Delta R}{\ln \Lambda} - 2 \frac{\Delta R}{\ln \Lambda} + 2. \]  

with a relative error of magnitude \( O(\sigma^2 c^2 / p^2) \) + \( O(k_0 / p) \), and where \( \Delta R = c^2 / 4 \pi \varepsilon_0 h c \simeq 1 / 137 \) is the fine-structure constant. We have introduced a bremsstrahlung logarithm \( \Delta R = \ln(k_0 / \Lambda) \), which arises in a way similar to the Coulomb logarithm \( \ln \Lambda \) for elastic collisions, and is due to cutting off the logarithmic divergence at some lowest photon energy \( k_c \). This
energy corresponds to photons emitted at the plasma frequency \( \omega_p \), at which point polarization of the background will dampen the bremsstrahlung interactions [15], and is thus given by \( k_c = \hbar \omega_p \). This gives a bremsstrahlung logarithm \( \ln \Lambda_B \approx 21 + \ln \left( \frac{k_0}{(m_e c^2 \sqrt{n_{20}})} \right) \), where \( n_{20} = n_e/(10^{20} \text{ m}^{-3}) \) is the electron density in units of \( 10^{20} \text{ m}^{-3} \). Assuming a plasma with \( \ln \Lambda = 15 \), \( n_{20} = 1 \) and choosing \( k_0 = 0.01 p \), the ratio (6) is of order 10% at 30 MeV, 50% at 2 GeV and 100% at 30 GeV, demonstrating that angular deflection caused by the emission of low-energy photons can contribute significantly to the motion of highly energetic electrons.

The bremsstrahlung collision operator has been implemented in the initial-value continuum kinetic-equation solver CODE (Collisional Distribution of Electrons) [9]. For this study we use CODE to solve the equation

\[
\frac{\partial f_e}{\partial t} - eE_\| \frac{\partial f_e}{\partial p_\|} = C_{\text{FP}} \{ f_e \} + C_B \{ f_e \},
\]

which in a magnetized plasma represents the gyroaveraged kinetic equation, with the parallel direction given by the magnetic field \( B \). The equation is also valid for an unmagnetized plasma which is cylindrically symmetric around the electric field \( E \). Elastic collisions are accounted for by the linearized relativistic Fokker-Planck operator for Coulomb collisions \( C_{\text{FP}} \), and \( C_B \) is the bremsstrahlung operator \( C_{\text{b}} \) summed over all particle species \( b \) in the plasma. Both thermal and fast electrons are resolved simultaneously, allowing runaway generation as well as the slowing-down of the fast population to be accurately modeled.

We will compare the effect of bremsstrahlung radiation losses on the momentum-space distribution of fast electrons using several models. The contribution from the emission of large-energy photons (with \( k > k_0 \)) are accounted for by either the Boltzmann operator in (3) or its approximation without angular deflection (4), while the low-energy photon contribution (\( k < k_0 \)) is described by (5). For the numerical solutions we choose an energy-dependent cut-off \( k_0 = m_e c^2(\gamma - 1)/1000 \). The Boltzmann models will be compared to the mean-force model where the bremsstrahlung losses are accounted for by an isotropic force term in the kinetic equation, defined as \( \mathbf{F}_B = -\mathbf{p} \sum_b m_b f_b m_e c^2(\gamma - 1) \frac{dk}{d\sigma_{eb}} \frac{dk}{\gamma} \), which is chosen to produce the correct average energy-loss rate [1].

To characterize the effect of bremsstrahlung on the electron distribution, we investigate quasi-steady-state numerical solutions of the kinetic equation (7). These are obtained by evolving the distribution function in time until an equilibrium is reached, typically after a few seconds at density \( n_{20} = 1 \) if an initial seed of fast electrons is provided. We investigate a range of electric-field values near the minimum field \( E_c = 4\pi \ln \Lambda_n r_i^2 m_e c^2/e \) to overcome collisional friction [4], using plasma parameters characteristic of tokamak-disruption experiments with massive gas injection.

![Figure 1](image-url)

**Figure 1** shows the electron distribution function in momentum space, calculated using CODE, with full Boltzmann bremsstrahlung effects included (black, solid); neglecting angular deflections in the large-\( k \) contribution (yellow, dash-dotted); also neglecting the small-\( k \) contribution (blue, dashed); and finally using the mean-force model (red, solid). Non-monotonic features (bumps) form in the mean-force as well as the Boltzmann models, but their characteristics are significantly different. With the Boltzmann models, an extended tail forms in the electron distribution. In contrast, the mean-force model produces a sharp feature, located where the energy gain due to the electric-field acceleration balances friction and bremsstrahlung losses. The addition of low-\( k \) scatterings (5), which lead to large-angle deflections, causes a subpopulation of fast electrons with significant perpendicular momentum to form. Furthermore, (3) and (4) appear to generally produce the same qualitative features, indicating that scatterings involving large-energy photons are well approximated by neglecting the angular deflection of the electron.

Inclusion of synchrotron radiation losses associated with the gyromotion of electrons in a straight magnetic field has been shown to be an important energy-loss mechanism [16–20]. Figure 1(b) shows that, in conjunction with bremsstrahlung losses, synchrotron losses (modelled as in [16]) shifts the distribution towards lower energies but does not change its qualitative features. The difference between the Boltzmann and mean-force models is therefore reduced in such cases, as the extent of
the distribution when full bremsstrahlung effects are included is reduced by the synchrotron effect.

Angle-averages of the electron distribution functions in figure 1 are shown as a function of electron kinetic energy $W = m_e c^2 (\gamma - 1)$. When there are no synchrotron losses present, the difference between the Boltzmann models for bremsstrahlung losses is insignificant. In the presence of effects which are sensitive to the angular distribution of electrons, such as synchrotron radiation losses (which are proportional to $p_z^2$), the difference is somewhat enhanced as angular deflection amplifies the dissipation.

To quantify the width in energy of the fast-electron tail, figure 3 shows the fraction of total plasma kinetic energy carried by electrons with energy greater than $W$, for a range of plasma compositions and electric fields. Again, the steady-state solutions are considered, and the energy ratio is calculated as $\int_0^\infty dW W (dn_e/dW)/W_{\text{tot}}$. When normalized to the energy $W_{\text{0}}$ which solves the energy-balance equation $\epsilon E_|| - \epsilon E_\perp + F_B = 0$ (accounting for collisional and bremsstrahlung energy loss), the behavior is seen to be insensitive to electric field and effective charge. The Boltzmann model consistently predicts that a fraction of the electron population reaches significantly higher energies than in the mean-force model, where all electrons have energy near $W_0$. For instance, in the Boltzmann model 5% of the plasma energy is carried by electrons with energy more than 2$W_0$.

Summary — We have developed a kinetic description of the effect of spontaneous bremsstrahlung emission on energetic electrons in plasmas. A computationally efficient representation of the bremsstrahlung collision operator has been obtained using an expansion in Legendre polynomials, with which the operator is reduced to a set of 1D energy integrals. This allows for rapid evaluation of the self-consistent electron distribution function in the presence of bremsstrahlung losses derived from the full Boltzmann operator.

By treating bremsstrahlung emission as a discrete process, we have shown that electrons may be accelerated to significantly higher energies than would be predicted by energy balance alone, with a significant fraction of particles reaching at least twice the expected energy. The explanation for this can be intuitively understood in the single-particle picture, where the new model allows some electrons to suddenly lose a large fraction of their energy in one emission, whereas others may be accelerated for a long time before a bremsstrahlung reaction occurs, thereby allowing higher maximum energies to be reached. This has important implications for the interpretation of experimental observation of fast electron beams in plasmas where bremsstrahlung losses are important, such as in magnetic-fusion plasmas. Furthermore, new effects are revealed in our treatment, as the emission of soft photons is found to contribute to angular deflection of the electron trajectory at a rate that increases with electron energy. This effect shifts part of the momentum-space distribution function towards higher perpendicular momenta, which in turn has implications for e.g. the destabilization of kinetic instabilities or the level of synchrotron radiation loss in magnetized plasmas.

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Paper B

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Kinetic modelling of runaway electrons in dynamic scenarios,
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Kinetic modelling of runaway electrons in dynamic scenarios

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Abstract. Improved understanding of runaway-electron formation and decay processes are of prime interest for the safe operation of large tokamaks, and the dynamics of the runaway electrons during dynamical scenarios such as disruptions are of particular concern. In this paper, we present kinetic modelling of scenarios with time-dependent plasma parameters; in particular, we investigate hot-tail runaway generation during a rapid drop in plasma temperature. With the goal of studying runaway-electron generation with a self-consistent electric-field evolution, we also discuss the implementation of a conservative collision operator and demonstrate its properties. An operator for avalanche runaway-electron generation, which takes the energy dependence of the scattering cross section and the runaway distribution into account, is investigated. We show that the simplified avalanche model of Rosenbluth & Putvinskii [1] can give very inaccurate results for the avalanche growth rate (either lower or higher) for many parameters, especially when the average runaway energy is modest, such as during the initial phase of the avalanche multiplication. The developments presented pave the way for an improved modelling of runaway-electron dynamics during disruptions or other dynamic events.

1. Introduction

Runaway electrons, a phenomenon made possible by the decrease of the collisional friction with particle energy [2], are common in plasmas in the presence of strong external electric fields or changing currents. The tightly focused beam of highly relativistic particles can be a serious threat to the first wall of a fusion reactor, due to the possibility of localized melting or halo current generation [3]. In the quest for avoidance or mitigation of the harmful effects of runaway-electron losses, a greater understanding of the runaway-electron phenomenon is required [4]. Improved knowledge of runaway-electron formation mechanisms, dynamics and characteristics will benefit the fusion community and contribute to a stable and reliable operation of reactor-scale tokamaks.

Kinetic simulation is the most accurate and useful method for investigating runaway-electron dynamics, and we recently developed a new tool called CODE (COllisional Distribution of Electrons [5]) for fast and detailed study of these
Kinetic modelling of runaway electrons in dynamic scenarios

processes. CODE solves the spatially homogeneous kinetic equation for electrons in 2-D momentum space, including electric-field acceleration, collisions, avalanche runaway generation and synchrotron-radiation-reaction losses [5, 6, 7]. In CODE, momentum space is discretized using finite differences in momentum and a Legendre-mode decomposition in pitch-angle cosine. Often, the time evolution of the distribution is the desired output, but a (quasi-)steady-state solution can also be efficiently obtained through the inversion of a single sparse system (in the absence of an avalanche source). CODE has been used to study the spectrum of the synchrotron emission emitted by runaways [5], the corresponding influence of the emission on the distribution function [6, 7, 8], and the factors influencing the critical electric field for runaway-electron generation [6, 9].

In this paper we describe improvements to CODE which enable us to investigate the effect of hot-tail runaway generation on the distribution (Section 2). This process can be the dominant mechanism in rapidly cooling plasmas. We also discuss the implementation of a full linearized collision operator, and demonstrate its conservation properties (Section 3). The use of this operator is necessary in cases where the correct plasma conductivity is required, and our implementation indeed reproduces the Spitzer conductivity [10] for weak electric fields. In addition, an improved model for the large-angle (knock-on) Coulomb collisions leading to avalanche multiplication of the runaway population [11], is described in Section 4. This model takes the energy dependence of the runaway distribution into account, and uses the complete energy-dependent Møller scattering cross section [12]. We find that its use can in some cases lead to significant modifications to the avalanche growth rate, compared to the more simplified model of Rosenbluth & Putvinskii [1].

The improvements described in this work enable the detailed study of runaway processes in dynamic situations such as disruptions, and the conservative collision operator makes self-consistent calculations of the runaway population and current evolution in such scenarios feasible [13].

2. Time-dependent plasma parameters

To be able to investigate the behavior of the electron population in dynamic scenarios such as disruptions or sawtooth crashes, it is necessary to follow the distribution function as the plasma parameters change. To this end, CODE has been modified to handle time-dependent background-plasma parameters. Since the kinetic equation is treated in linearized form, the actual temperature and density of the distribution are determined by the background Maxwellian used in the formulation of the collision operator. This allows for a scheme where the kinetic equation is normalized to a reference temperature \( \tilde{T} \) and number density \( \tilde{n} \), so that the discretized equation can be expressed on a fixed reference grid in momentum space. (Throughout this paper, we will use a tilde to denote a reference quantity.) By changing the properties of the Maxwellian equilibrium around which the collision operator is linearized, plasma-parameter evolution can be modelled
on the reference grid without the need for repeated interpolation of the distribution function to new grids.

Analogously to Ref. [5], the kinetic equation in 2D momentum space for the electron distribution function $f$ experiencing an electric field $E$ (parallel to the magnetic field) and collisions, can be expressed as

$$\frac{\partial F}{\partial t} + \hat{E} \left( \xi \frac{\partial F}{\partial y} + \frac{1 - \xi^2}{y} \frac{\partial F}{\partial \xi} \right) = \hat{C} \{F\} + \hat{S} \{F\}.$$  \hfill (1)

Here we have introduced a convenient normalized momentum $y = \gamma v/\bar{v}_e$ – where $\bar{v}_e = \sqrt{2T/m}$ is the reference electron thermal speed – and the cosine of the pitch angle $\xi = y_\parallel/y$. Using $\kappa = m^3 \bar{v}_e^3 \pi^{3/2}/\bar{v}$, we have also defined the distribution function $F = F(y, \xi) = \kappa f$ (normalized so that $F(y = 0) = 1$ for a Maxwellian with $T = \bar{T}$ and $n = \bar{n}$), time $t = \bar{\nu}_{ee} t$, and electric field $\hat{E} = -e E/m \bar{v}_e \bar{\nu}_{ee}$, as well as the normalized operators $\hat{C} = C \kappa/\bar{v}_e$ and $\hat{S} = S \kappa/\bar{v}_e$, with $\bar{\nu}_{ee} = 16\sqrt{\pi} \bar{e} \bar{n} \ln \bar{\Lambda}/3m^2 \bar{v}_e^3$ the reference electron thermal collision frequency, $-e$, $m$ and $v$ the charge, rest mass and speed of the electron, and $\gamma$ the relativistic mass factor. Note that $|\hat{E}| = (3\sqrt{\pi}/4) E/E_D$, with $E_D$ the Dreicer field [2]. $C$ is the Fokker-Planck collision operator and $S$ an operator describing close (large-angle) Coulomb collisions. These operators will be discussed more thoroughly in Sections 3 and 4, respectively; for now we just state the formulation of the collision operator employed in Ref. [5] using the normalizations above:

$$\hat{C}^{tp} = c_C \bar{v}_e^3 y^{-2} \left( \frac{\partial}{\partial y} \left[ y^2 \Psi \left( \frac{1}{x} \frac{\partial}{\partial y} + \frac{2}{\bar{v}_e^2} \right) F \right] + \frac{c_\xi}{2x} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial F}{\partial \xi} \right).$$ \hfill (2)

The superscript $\text{tp}$ denotes that this is the test-particle part of the linearized collision operator $C^t$ discussed in Section 3. Here (and throughout the rest of this paper), a bar denotes a quantity normalized to its reference value (i.e, $\bar{v}_e = v_e/\bar{v}_e$, $x = y/\gamma = v/\bar{v}_e$ is the normalized speed, $c_C = 3\sqrt{\pi} \bar{v}_{ee}/4$, $c_\xi = Z_{\text{eff}} + \Phi - \Psi + \bar{v}_e^2 \delta x/2$, $Z_{\text{eff}}$ is the effective ion charge, $\Phi = \Phi(x/\bar{v}_e)$ and $\Psi = \Psi(x/\bar{v}_e) = \bar{v}_e^2[\Phi - \bar{v}_e^{-1} x d\Phi/d(x/\bar{v}_e)]/2x^2$ are the error and Chandrasekhar functions, respectively, and $\delta = \bar{v}_e/c$ (with $c$ the speed of light) is assumed to be a small parameter (i.e the thermal population is assumed to be non-relativistic).

Changes to the plasma temperature manifest as shifts in the relative magnitude of the various terms in Eq. (2) (through $\delta$ and the quantities with a bar), as well as a change in the overall magnitude of the operator, whereas changes in density only have the latter effect. In both cases, the distribution is effectively colliding with (and relaxing towards) a Maxwellian different from the one native to the reference momentum grid. Heat or particles are introduced to (or removed from) the bulk of the distribution when using this scheme, as all changes to plasma parameters are described by changes to the Maxwellian. This provides a powerful way of simulating rapid cooling, for instance associated with a tokamak disruption.
2.1. Hot-tail runaway-electron generation

If the time scale of the initial thermal quench in a disruption event is short enough – comparable to the collision time – the tail of the initial Maxwellian electron distribution will not have time to equilibrate as the plasma cools. The particles in this supra-thermal tail may constitute a powerful source of runaway electrons, should a sufficiently strong electric field develop before they have time to reconnect with the bulk electrons. This process is known as hot-tail generation, and can be the dominant source of runaways under certain conditions [14, 15]. It has previously been investigated analytically or using Monte-Carlo simulations [15, 16] or purpose-built finite-difference tools [16, 17]. Using CODE to model a temperature drop enables the efficient study of a wider range of scenarios, and allows full use of other capabilities of CODE, such as avalanche generation or synchrotron radiation reaction. Here, we restrict ourselves to a proof-of-principle demonstration, and leave a more extensive investigation to future work.

To facilitate a comparison to the theoretical work by Smith and Verwichte [17], we will model a rapid exponential temperature drop, described by

\[ T(t) = T_i + (T_0 - T_i)e^{-t/t_*}, \]

with \( T_0 = 3.1 \text{ keV} \) the initial temperature, \( T_i = 31 \text{ eV} \) the final temperature, and \( t_* = 0.3 \text{ ms} \) the cooling time scale. We also include a time-dependent electric field described by

\[ \frac{E(t)}{E_D} = \left( \frac{E}{E_D} \right)_0 \sqrt{\frac{T_0}{T(t)}}, \]

with \( (E/E_D)_0 = 1/530 \) the initial normalized electric field. The temperature and electric-field evolutions are shown in Fig. 1a and are the same as those used in Fig. 5 of Ref. [17], as are all other parameters in this section.

Figure 1b, in which the additional parameters \( n = 2.8 \cdot 10^{19} \text{ m}^{-3} \), and \( Z_{\text{eff}} = 1 \) were used, illustrates the distribution-function evolution during the temperature drop.
The figure shows that as the temperature decreases, most of the electrons quickly adapt. At any given time \( t \), the bulk of the distribution remains close to a Maxwellian corresponding to the current temperature \( T(t) \). The initially slightly more energetic electrons, although part of the original bulk population, thermalize less efficiently. On the short cooling time-scale, they remain as a distinct tail, and as the thermal speed decreases they become progressively less collisional. This process is evident in the first three time steps shown (\( t = 0.025 \text{–} 0.83 \text{ ms} \)). In the final time step, the electric field has become strong enough to start to affect the distribution, and a substantial part of the high-energy tail is now in the runaway region. This can be seen from the qualitative change in the tail of the distribution, which now shows a positive slope associated with a strong flow of particles to higher momenta.

For the temperature evolution in Eq. (3), analytical results for the hot-tail runaway generation were obtained in Ref. [17]. Assuming the background density to be constant, the runaway fraction at time \( t \) can be written as

\[
\frac{n_{r,\text{dir}}}{n} = \frac{4}{\sqrt{\pi}} \int_{\infty}^{\infty} \left[ 1 - \frac{(u_c^3 - 3\tau)^{2/3}}{(u^3 - 3\tau)^{2/3}} \right] e^{-u^2} u^2 \, du,
\]

where \( \tau(t) = (3\sqrt{\pi}/4) \nu_{ee}(t-t_\star) = (3\sqrt{\pi}/4)(\hat{t} - \hat{t}_\star) \) is a normalized time, \( u(t) = x^{[0]} + 3\tau(t) \), \( x^{[0]} \) is the speed normalized to the initial thermal speed, and \( u_c \) is related to the critical speed for runaway generation: \( u_c(t) = x_c^{[0]} + 3\tau(t) \). Equation (5), which corresponds to Eq. (18) in Ref. [17], is only valid when a significant temperature drop has already taken place (as manifested by the appearance of the cooling time scale \( t_\star \) as a "delay" in the expression for \( \tau \), see [17]). Equation (5) is derived in the absence of an electric field; only an exponential drop in the bulk temperature is assumed. The electric field shown in Fig. 1a is only used to define a runaway region, so that the runaway fraction can be calculated. In other words, it is assumed that the electric field does not have time to influence the distribution significantly during the temperature drop.

The runaway fraction calculated using Eq. (5) includes only the electrons in the actual runaway region, i.e. particles whose trajectories (neglecting collisional momentum-space diffusion) are not confined to a region close to the origin. In this case, the lower boundary of the runaway region is given in terms of the limiting (non-relativistic) momentum \( y \) for a given \( \xi \): \( y \geq y_c \xi = (\delta^2[\xi + 1]E/2E_c - 1)^{-1/2} \) [16], where \( E_c = 4\pi e^3 n \ln \Lambda/mc^2 \) is the critical electric field for runaway generation [18]. The temperature drop does however lead to an isotropic high-energy tail (in the absence of an electric field). By defining the runaway region as \( y > y_c = (\delta^2[\xi + 1]E/2E_c - 1)^{-1/2} \), thereby including all particles with \( v > v_c \), Eq. (5) can be simplified to

\[
\frac{n_{r}}{n} = \frac{2}{\sqrt{\pi}} u_c e^{-u_c^2} + \text{erfc}(u_c),
\]

where \( \text{erfc}(x) \) is the complementary error function. By default, CODE uses such an isotropic runaway region, which is a good approximation in the case of only Dreicer and avalanche generation (especially once the runaway tail has become substantial); however, in the early stages of hot-tail-dominated scenarios, the isotropic runaway region
Figure 2. Hot-tail runaway density obtained using CODE – with (blue, dashed) and without (yellow, dash-dotted; red, dotted) an electric field included during the temperature drop – and the analytical estimates Eqs. (5) and (6) (black, solid), for the temperature and E-field evolution in Fig. 1a. An a) ξ-dependent and b) isotropic lower boundary of the runaway region was used. The collision operator in Eq. (2) was used for the blue and yellow lines, whereas its non-relativistic limit was used for the red and black lines.

significantly overestimates the actual runaway fraction, and the lower boundary \( y_{c\xi} \) must be used.

Figure 2 compares the runaway density evolution computed with CODE, using both ξ-dependent and isotropic runaway regions, to Eqs. (5) and (6), respectively. The parameters of the hot-tail scenario shown in Fig. 1 were used, and no avalanche source was included in the calculation. The collision operator used in Ref. [17] is the non-relativistic limit of Eq. (2), with \( c_\xi = 0 \) (since the distribution is isotropic in the absence of an electric field). CODE results using both this operator (red, dotted) and the full Eq. (2) (yellow, dash-dotted) are plotted in Fig. 2, with the latter producing \( \sim 50\% \) more runaways in total. This difference can likely be explained by the relatively high initial temperature (3 keV) in the scenario considered, in which case the non-relativistic operator is not strictly valid for the highest energy particles. Good agreement between CODE results and Eqs. (5) and (6) (black, solid) is seen for the saturated values in the figure. A CODE calculation where the electric-field evolution is properly included in the kinetic equation (corresponding to the distribution evolution in Fig. 1b) is also included (blue, dashed), showing increased runaway production. In the isotropic case (Fig. 2b), the increase is smaller than a factor of 2, and neglecting the influence of the electric field can thus be considered reasonable for the parameters used, at least for the purpose of gaining qualitative understanding. With the ξ-dependent runaway region (Fig. 2a), the change in runaway generation is more pronounced, and the inclusion of the electric field leads to an increase by almost an order of magnitude. Note that the final runaway density with the electric field included is very similar in Figs. 2a and 2b, indicating that the details of the lower boundary of the runaway region become unimportant once the tail is sufficiently large. Throughout the remainder of this paper we will make use of the isotropic runaway region.

We conclude that, in order to obtain quantitatively accurate results, the electric
field should be properly included, and a relativistic collision operator should be used. This is especially true when modelling ITER scenarios, where the initial temperature can be significantly higher than the 3 keV used here.

3. Conservative linearized Fokker-Planck collision operator

Treating the runaway electrons as a small perturbation to a Maxwellian distribution function, the Fokker-Planck operator for electron-electron collisions [19, 20] can be linearized and written as \( C\{f\} \simeq C^l\{f\} = C^{tp} + C^{fp} \). The so-called test-particle term, \( C^{tp} = C^{nl}\{f_1, f_M\} \), describes the perturbation colliding with the bulk of the plasma, whereas the field-particle term, \( C^{fp} = C^{nl}\{f_M, f_1\} \), describes the reaction of the bulk to the perturbation. Here \( C^{nl} \) is the non-linear Fokker-Planck-Landau operator, \( f_M \) denotes a Maxwellian, and \( f_1 = f - f_M \) the perturbation to it (\( f_1 \ll f_M \)). Collisions described by \( C\{f_1, f_1\} \) are neglected since they are second order in \( f_1 \). The full linearized operator \( C^l \) conserves particles, momentum and energy. Since it is proportional to a factor \( \exp(-y^2) \), the field-particle term mainly affects the bulk of the plasma, and is therefore commonly neglected when studying runaway-electron kinetics. The test-particle term in Eq. (2) only ensures the conservation of particles, however, not momentum or energy.

Under certain circumstances, it is necessary to use a fully conservative treatment also for the runaway problem, in particular when considering processes where the conductivity of the plasma is important. In the study of runaway dynamics during a tokamak disruption using a self-consistent treatment of the electrical field, accurate plasma current evolution is essential, and the full linearized collision operator must be used. A non-linear collision operator valid for arbitrary particle (and bulk) energy has been formulated [21, 22]. The collision operator originally implemented in CODE is the result of an asymptotic matching between the highly relativistic limit of the test-particle term of the linearized version of that operator, with the usual non-relativistic test-particle operator [23], and is given in Eq. (2). The relativistic field-particle term is significantly more complicated, however, and its use would be computationally more expensive. Here we instead implemented the non-relativistic field-particle term, as formulated in Refs. [24, 25]. As will be shown, this operator (together with the non-relativistic limit of Eq. 2) accurately reproduces the Spitzer conductivity for sufficiently weak electric fields and temperatures where the bulk is non-relativistic. Using the normalization in Section 2, the field-particle term is

\[
\dot{C}^{fp} = \frac{\epsilon_C}{\sqrt{\pi}y^2 e^{-\frac{\epsilon}{\epsilon_e}} x^2} \left[ \frac{2x^2}{\tilde{v}_e^4} \frac{\partial^2 G}{\partial x^2} - \frac{2}{\tilde{v}_e^2} H + 4\pi F \right],
\]

where \( G \) and \( H \) are the Rosenbluth potentials, obtained from the distribution using

\[
\tilde{v}_e^2 \nabla_x^2 H = -4\pi F, \quad \tilde{v}_e^2 \nabla_x^2 G = 2H.
\]

The system of equations composed of Eqs. (7-8), together with the non-relativistic limits of Eqs. (1-2) \( (y \rightarrow x \text{ and } \delta \rightarrow 0) \), is discretized (see Ref.[5]) and solved using an efficient
Kinetic modelling of runaway electrons in dynamic scenarios

Figure 3. a) Parallel momentum and b) energy moments of the distribution function in CODE, using different collision operators. Initially, \( E = 50 \text{ V/m} \) and \( Z_{\text{eff}} = 1 \) were used, but for \( t > t_0 \), the electric field was turned off and the ion charge set to \( Z_{\text{eff}} = 0 \). Using two Legendre modes for the field-particle term was sufficient to achieve good conservation of energy and parallel momentum.

Method described in Ref. [26]. The equations are combined into one linear system of the form

\[
\begin{pmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & 0 \\
0 & M_{31} & M_{33}
\end{pmatrix}
\begin{pmatrix}
F \\
G \\
H
\end{pmatrix}
= 
\begin{pmatrix}
S_i \\
0 \\
0
\end{pmatrix},
\]

where the first row describes the kinetic equation (1) (with \( S_i \) representing any sinks or sources), and the second and third rows correspond to Eq. (8). This approach makes it possible to consistently solve for both the Rosenbluth potentials and the distribution with a single matrix operation. Since there is no explicit need for the Rosenbluth potentials, however, \( G \) and \( H \) can be eliminated by solving the block system analytically:

\[
(M_{11} - \left[M_{12} - M_{13} M_{33}^{-1} M_{32}\right] M_{22}^{-1} M_{21}) F \equiv MF = S_i,
\]

If only the test-particle operator (Eq. 2) is used, \( M \) reduces to \( M_{11} \). Since the Rosenbluth potentials are defined through integrals of the distribution, the field-particle term introduces a full block for each Legendre mode into the normally sparse matrix describing the system. However, the integral dependence on \( F \) also implies that significantly fewer modes are required to accurately describe the potentials (compared to \( l^2 \)), and the additional computational cost is modest. (The operator \( \nabla_l^2 \) is proportional to \( l^2 \), with \( l \) the Legendre mode index, and \( G \) and \( H \) therefore decay rapidly with increasing \( l \).)

The conservation properties of the full non-relativistic collision operator (Eqs. 2 and 7), as well as the relativistic test-particle operator in Eq. (2), are shown in Fig. 3. As an electric field is applied to supply some momentum and energy to the distribution, the parallel momentum (Fig. 3a) quickly reaches a steady-state value corresponding to the plasma conductivity, which differs by about a factor of two for the two operators (see below). The electric field is turned off at \( t = t_0 = 100 \) collision times (and \( Z_{\text{eff}} = 0 \) is imposed to isolate the behavior of the electron-electron collision operator), at which point the parallel momentum for the operator in Eq. (2) (blue, dashed) is lost on a
short time scale as the distribution relaxes back towards a Maxwellian. In contrast, the full linearized operator (black, solid) conserves parallel momentum in a pure electron plasma, as expected.

The electric field continuously does work on the distribution, a large part of which heats the bulk electron population, but the linearization of the collision operator breaks down if the distribution deviates too far from the equilibrium solution. As long as a non-vanishing electric field is used together with an energy conserving collision operator, an adaptive sink term removing excess heat from the bulk of the distribution must be included in Eq. (1) to guarantee a stable solution. Physically this accounts for loss processes that are not included in the model, such as line radiation, bremsstrahlung and radial heat transport. The magnitude of the black line in Fig. 3b therefore reflects the energy content of the runaway population – not the total energy supplied by the electric field – since a constant bulk energy is enforced. The energy sink is not included for $t > t_0$ (since $E = 0$), however, and the energy conservation observed is due to the properties of the collision operator itself. Again, the use of the collision operator in Eq. (2) is associated with a quick loss of kinetic energy as soon as the electric field is removed.

The electrical conductivity of a fully ionized plasma subject to an electric field well below the Dreicer value – the Spitzer conductivity – can be expressed as

$$\sigma_S = L(Z_{\text{eff}}) \frac{ne^2}{Z_{\text{eff}} m v_{ce}},$$

(11)

where $L(Z_{\text{eff}})$ is a transport coefficient which takes the value $L \simeq 2$ in a pure hydrogen plasma [10]. Figure 4 demonstrates that the conductivity calculated with CODE reproduces the Spitzer value for moderate electric field strengths, if the conservative collision operator is used, and the initial Maxwellian adapts to the applied electric field on a time scale of roughly 10 collision times. For field strengths significantly larger than $E_c$, the conductivity starts to deviate from $\sigma_S$, as a runaway tail begins to form (Fig. 4b); in this regime, the calculation in Ref. [10] is no longer valid. Using the collision operator in Eq. (2) consistently leads to a conductivity which is lower by about a factor of 2, as expected (see for instance Ref. [27]). The runaway growth is also affected, with the conserving operator leading to a larger runaway growth rate.

4. Improved operator for knock-on collision

The Fokker-Planck collision operators discussed in Section 3 accurately describe grazing collisions – small-angle deflections which make up the absolute majority of particle interactions in the plasmas under consideration. Large-angle collisions are usually neglected as their cross section is significantly smaller, but in the presence of runaway electrons they can play an important role in the momentum space dynamics, as an existing runaway can transfer enough momentum to a thermal electron in one collision to render it a runaway, while still remaining in the runaway region itself. Such knock-on collisions can therefore lead to an exponential growth of the runaway density – an avalanche [28, 1].
Figure 4. a) Conductivity (normalized to the Spitzer value) and b) normalized runaway density, as functions of time for different collision operators (non-relativistic full linearized: solid, relativistic test-particle: dashed) and E-field strengths ($E/E_D = 1\%$: yellow, $E/E_D = 5\%$: red, $E/E_D = 6\%$: blue), considering only Dreicer runaway generation. The parameters $T=1$ keV, $n = 5 \cdot 10^{19} \text{m}^{-3}$ and $Z_{\text{eff}} = 1$ were used.

In the absence of a complete solution to the Boltzmann equation, we model avalanche runaway generation using an additional source term in the kinetic equation (1), evaluated for $y > y_c$. A commonly used operator was derived by Rosenbluth and Putvinski [1] and takes the form

$$\hat{S}_{\text{RP}} = \frac{n_r}{\bar{n}} n^2 \left[ \frac{3\pi \delta^3}{16 \ln \Lambda} \delta_D(\xi - \xi_2) \frac{1}{y^2} \frac{\partial}{\partial y} \left( \frac{1}{1 - \sqrt{1 + \delta^2 y^2}} \right) \right], \quad (12)$$

where $n_r$ is the number density of runaway electrons, $\bar{n}$ is the density normalized to its reference value, and $\delta_D$ is the Dirac $\delta$-function. In the derivation, the momentum of the incoming particle is assumed to be very large (simplifying the scattering cross section) and its pitch-angle vanishing ($\xi = 1$). It is also assumed that the incoming particle is unaffected by the interaction. This implies that the generated secondary particles are all created on the curve $\xi = \xi_2 = \delta y / (1 + \sqrt{1 + \delta^2 y^2})$ (which is a parabola in $[y_\parallel, y_\perp]$-space), and that all runaways (from the point of view of the avalanche source) are assumed to have momentum $p = \gamma v/c = \delta y \gg 1$ (since $\hat{S}_{\text{RP}} \propto n_r$). They can therefore contribute equally strongly to the avalanche process. This has the peculiar and non-physical consequence that particles can be created with an energy higher than that of any of the existing runaways. The $\delta$-function in $\xi$ is numerically ill-behaved, as it produces significant oscillations (Gibbs phenomenon) when discretized using the Legendre-mode decomposition employed in CODE (see Fig. 5a).

An operator that relaxes the assumption of very large runaway momentum has been presented by Chiu et al. [11]. It has the form

$$\hat{S}_{\text{Ch}}(y, \xi) = \bar{n} \frac{2\pi e^4}{m^2 c^3} \frac{\bar{n} e^4}{\nu_{ee}} \frac{x}{y^2 \xi} (y_{in})^4 F^*(y_{in}) \Sigma(\gamma, \gamma_{in}), \quad (13)$$

where

$$\Sigma(\gamma, \gamma_{in}) = \frac{\gamma_{in}^2}{(\gamma_{in}^2 - 1)(\gamma - 1)^2(\gamma_{in} - \gamma)^2} \left[ (\gamma_{in} - 1)^2 \right]$$
Kinetic modelling of runaway electrons in dynamic scenarios

![Figure 5. Contour plots of the magnitude of the source in a) Eq. (12) and b) Eq. (13) in \((y_\parallel, y_\perp)\) momentum space, given the same electron distribution. The plotted quantity is \(\log_{10} \hat{S}\) and \(y_c\) defines the lower bound of the runaway region. The angle-averaged source magnitudes are shown in c). The parameters \(T = 1\) keV, \(n = 5 \times 10^{19}\) m\(^{-3}\), \(Z_{\text{eff}} = 1\) and \(E = 1\) V/m, with \(\max(y) = 70\), were used to obtain the distribution, and the simulation was run for 300 collision times with primary generation only.](image)

\[
\xi = \frac{(\gamma - 1)(\gamma_{\text{in}} - 1)}{\gamma_{\text{in}} - 1},
\]

which restricts the region where the source is non-vanishing. (This relation is analogous to the parabola \(\xi_2\) in the case of the operator in Eq. 12.) Since the electrons participating in a collision are indistinguishable, it is sufficient to consider only the cases where the energy of the created secondary runaway is less than half of the primary energy, \((\gamma - 1) \leq (\gamma_{\text{in}} - 1)/2\), which with the above equation leads to the condition \(\xi \leq \xi_{\text{max}} = \sqrt{\gamma/(\gamma + 1)}\). By the same argument, the maximum attainable runaway energy in the simulation (the maximum of the momentum grid) leads to the condition \(\xi \geq \xi_{\text{min}} = \sqrt{(\gamma - 1)(\gamma_{\text{max}} + 1)/(\gamma + 1)(\gamma_{\text{max}} - 1)}\).

The magnitudes of the two sources (12) and (13) are computed from a given typical runaway distribution function, and shown in Fig. 5a and b. Curves corresponding to the parabola \(\xi_2\), as well as the limits \(\xi_{\text{min}}\) and \(\xi_{\text{max}}\) are also included. Note that the amount of numerical noise is significantly reduced for the source in Eq. (13). In order to avoid double-counting the small-angle collisions described by the Fokker-Planck-Landau
collision operator $C$, the knock-on source must be cut off at some value of momentum sufficiently far from the thermal bulk. As can be seen from the figure, however, the magnitude of both sources increases with decreasing momenta, and the avalanche growth rate is therefore sensitive to the specific choice of momentum cut-off. Since our particular interest is the generation of runaway electrons, we choose to place the cut-off at $y = y_c$, so that the sources are non-vanishing only in the runaway region [5, 14]. Secondary particles deposited just below the threshold – although not technically runaways – could eventually diffuse into the runaway region, thereby potentially increasing the Dreicer growth rate. In Ref. [29], such effects were however shown to be negligible for the operator in Eq. (12), indicating that the vast majority of particles deposited at $y < y_c$ are slowed down rather than accelerated (as expected). This reduces the sensitivity of the avalanche growth rate to the choice of momentum cut-off (as long as $y_{\text{cut-off}} \leq y_c$), and reaffirms our choice $y_{\text{cut-off}} = y_c$.

Fig. 5c shows the source magnitudes integrated over pitch-angle, and as expected, the source in Eq. (13) extends only up to $y \simeq y_{\text{max}}/2$, whereas the source in Eq. (12) is non-vanishing also for larger momenta. The amount of secondary runaways generated by the two sources agree well at low energies, but less so further away from the bulk. In this particular case, the total source magnitude $\int \hat{S} y^2 dy d\xi$ agrees to within 25%, as most of the secondaries are created close to the boundary of the runaway region.

### 4.1. Avalanche growth rates for the different operators

In general, the avalanche growth rate produced by the two sources can differ substantially. We will illustrate this point by considering the Møller cross section in more detail. We choose to quantify the source magnitude for an arbitrary distribution by computing the cross section, integrated over the energy of the outgoing (secondary) particle and normalized to $r_0^2$, with $r_0$ the classical electron radius. In other words, we look at the total normalized cross section for an incoming particle with $\gamma_{\text{in}}$ to participate in a knock-on collision resulting in avalanche [30]:

$$K_{\text{Ch}}(\gamma_{\text{in}}) = \int_{\gamma_c}^{\infty} \frac{(\gamma_{\text{in}} - 1)^{1/2 + 1}}{\Sigma(\gamma, \gamma_{\text{in}})} d\gamma$$

$$= (\gamma_{\text{in}}^2 - 1)^{-1} \left[ \frac{\gamma_{\text{in}}^2}{\gamma_c - 1} + \frac{\gamma_{\text{in}}^2}{\gamma_c - \gamma_{\text{in}}} \right.$$  

$$\left. + 2\gamma_{\text{in}} - 1 \ln \left( \frac{\gamma_c - 1}{\gamma_{\text{in}} - \gamma_c} \right) + \frac{\gamma_{\text{in}} + 1}{2} - \gamma_c \right] , \quad (16)$$

where $\gamma_c = \sqrt{(E/E_c)/(E/E_c - 1)}$ corresponds to the critical momentum for runaway generation and the upper integration boundary stems from the condition leading to $\xi_{\text{max}}$. This expression is relevant to the source in Eq. (13), which uses the complete cross section (14), whereas for the more simple source in Eq. (12), only the leading order term in $\gamma_{\text{in}}$ in the scattering cross section is taken into account. This corresponds
Kinetic modelling of runaway electrons in dynamic scenarios

Figure 6. a) Contours (black, white) of the ratio of total cross-sections ($K_{Ch}/K_{RP}$) for an electron with $p_{in}$ to contribute to the avalanche process, as a function of $p_{in} = \gamma_{in} v_{in}/c = \sqrt{\gamma_{in}^2 - 1}$ and $E/E_c$. b) Ratio of avalanche growth rates ($\Gamma_{Ch} - \Gamma_D$)/$(\Gamma_{RP} - \Gamma_D)$ in CODE simulations. The parameters $T \in [0.1 \text{ eV}, 5 \text{ keV}]$, $E/E_c \in [1.1, 1000]$, $n = 5 \cdot 10^{19} \text{ m}^{-3}$ and $Z_{eff} = 1$ were used.

to taking the high-energy limit of the above equation, so that

$$K_{RP} = \frac{1}{\gamma_c - 1}$$

becomes a simple constant.

To systematically explore the relative magnitude of the two sources, the ratio $K_{Ch}/K_{RP}$ is plotted in Fig. 6a. As expected, the two expressions agree very well at high primary momenta. At somewhat lower momenta, of the order $\gamma \approx p \approx 5$, two distinct regions are discernible. For $E/E_c \approx 10$ (the orange region), the simplified cross section is larger than the full expression, and the Rosenbluth-Putvinski operator (12) is likely to overestimate the avalanche generation. For $E/E_c \approx 10$, the opposite is true, and the operator in Eq. (13) has a significantly larger cross section for $E/E_c \approx 30$ (the blue region). The more accurate operator (13) should thus be expected to produce more runaways when the runaway population is at predominantly low energies, and $E/E_c$ is large. For both of these conditions to be fulfilled simultaneously (and at the same time avoid a slide-away scenario), the temperature must be low so that $E/E_D \ll 1$ even for large $E/E_c$. The effect is also likely to be most apparent at relatively early times, before the runaway tail has extended to multi-MeV energies.

CODE simulations support the above conclusions and show excellent qualitative agreement, as shown in Fig. 6b. The figure shows the ratio of final avalanche growth rates ($\Gamma_{Ch} - \Gamma_D$)/$(\Gamma_{RP} - \Gamma_D)$, with $\Gamma_i = n_{r}^{-1}(dn_r/dt)$ the growth rate obtained in a CODE run using source $i$ (here the subscript D denotes pure Dreicer generation). Each marker in the figure is thus computed from three separate CODE runs. As a proxy for $p_{in}$, the average runaway momentum $p_{r,av}$ in the final time step $t_f$ of the simulation without a source was used, and for a given $E/E_c$, different $p_{r,av}$ were obtained from simulations with varying values of $T$ (and corresponding values of $E/E_D$). The simulations were run for $t_{max} = 5000$ collision times, and $t_f$ was set to either $t_{max}$, the first time step for which
Figure 7. Ratio of avalanche growth rates $(\Gamma_{\text{Ch}} - \Gamma_{\text{D}})/(\Gamma_{\text{RP}} - \Gamma_{\text{D}})$ in CODE simulations, as a function of temperature. The same parameters as in Fig. 6 were used.

Figure 7 shows all the data points in Fig. 6b, as a function of temperature. The figure confirms that the region where the more accurate operator produces a significantly higher growth rate is only accessible at temperatures $T < 100$ eV (in the domain of validity of a linearized treatment). As is evident in the figure, however, regions where the Rosenbluth-Putvinski operator significantly overestimates the avalanche growth rate (points below 1 on the vertical axis) are present at all temperatures. The operator in Eq. (13) is thus of general interest.

Since the electric field spike responsible for the acceleration of runaways during a tokamak disruption is induced by the temperature drop, and therefore occurs slightly later than the drop itself, the temperature is low during the majority of the acceleration process. For significant runaway acceleration, $E/E_c \gg 1$ is therefore required, and during the initial part of the acceleration process, parameters are likely those corresponding to the blue region of Fig. 6a, where the improved avalanche source produces a significantly higher growth rate than the Rosenbluth-Putvinski operator. Post-thermal-quench temperatures in ITER are expected be as low as 10 eV and peak electric fields in disruptions can reach 80 V/m or more [31]. Towards the end of the thermal quench,
the normalized electric field is then $E/E_c \approx 1300$ (with $E = 80 \text{ V/m}$, $T = 50 \text{ eV}$ and $n = 1 \cdot 10^{20} \text{ m}^{-3}$). A typical ITER disruption would thus (at least initially) be firmly in the blue region of Fig. 6b, and the avalanche growth should be significantly higher than what the Rosenbluth-Putvinski source predicts. As the temperature is low, the runaways will also spend a comparatively long time at low momenta ($p \ll 1$), where the disagreement between the operators is most pronounced. Note that, according to the figure, an average runaway energy of several MeV ($p > 5 - 10$) is needed for the difference between the growth rates to become small for all $E/E_c$, at which point the most energetic electrons will have reached energies of several tens of MeV or more. However, since the electric field changes rapidly, the runaways may experience parameters corresponding to both the orange and blue regions in Fig. 6b before reaching such energies. Further work is therefore needed to assess the overall impact on the avalanche growth of using the improved operator (13), although it is clear that its use is essential for accurate analysis.

5. Summary

Runaway electrons are intimately linked to dynamic scenarios, as they predominantly occur during disruptions and sawtooth events in tokamaks. An accurate description of their dynamics in such scenarios requires kinetic modelling of rapidly changing plasma conditions, and mechanisms such as hot-tail runaway generation add to the already interesting set of phenomena of importance to the evolution of the runaway population.

In this paper we have described the modelling of several such processes, using the numerical tool CODE to calculate the momentum space distribution of runaway electrons. In particular, we have investigated rapid-cooling scenarios where hot-tail runaway-electron generation is dominant. Good agreement with previous theoretical work was observed, but CODE simulations also allow for flexible study of a variety of parameter regimes not readily accessible in analytical treatments, and involving other processes such as avalanche generation or synchrotron radiation.

Furthermore, the full linearized non-relativistic Fokker-Planck-Landau collision operator was discussed, and its implementation described. The operator was found to reproduce the expected Spitzer conductivity in the relevant parameter regime and showed excellent conservation properties. The use of such an operator is essential for the correct current evolution in self-consistent modelling, and in particular when studying the interplay between current and electric field evolution and runaway-electron generation during a disruption.

The process of avalanche multiplication of the runaway population via close Coulomb collisions was also considered, and an improved operator, relaxing some of the approximations of the commonly used Rosenbluth-Putvinski operator, was discussed. It was found that the avalanche growth rate can be significantly affected – increased for low temperatures and high $E/E_c$ and decreased for low $E/E_c$ – by the use of the new operator. The change to the growth rate can be especially large during the early stages of the runaway acceleration process, thus potentially affecting the likelihood of a
given runaway seed transforming into a serious runaway beam, and use of the improved operator is of particular relevance in disruption scenarios.

The work presented in this paper paves the way for a better understanding of runaway electron dynamics in rapidly changing scenarios, for instance during tokamak disruptions. It enables more accurate assessment of the risks posed by runaway electrons in situations of experimental interest, particularly in view of future tokamaks such as ITER.

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References

Paper C

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Radiation reaction induced non-monotonic features in runaway electron
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Radiation reaction induced non-monotonic features in runaway electron distributions

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Runaway electrons, which are generated in a plasma where the induced electric field exceeds a certain critical value, can reach very high energies in the MeV range. For such energetic electrons, radiative losses will contribute significantly to the momentum space dynamics. Under certain conditions, due to radiative momentum losses, a non-monotonic feature – a ‘bump’ – can form in the runaway electron tail, creating a potential for bump-on-tail-type instabilities to arise. Here, we study the conditions for the existence of the bump. We derive an analytical threshold condition for bump appearance and give an approximate expression for the minimum energy at which the bump can appear. Numerical calculations are performed to support the analytical derivations.

1. Introduction

In a plasma, the drag force from Coulomb collisions acting on fast electrons decreases with the electron velocity. Thus, if the electric field $E$ exceeds a threshold value $E_c$, electrons with sufficient velocity will be indefinitely accelerated and are called runaway electrons. The critical field $E_c$ is defined as

$$E_c = \frac{n_e e^3 \ln \Lambda}{4 \pi \varepsilon_0 m_e c^2},$$

where $n_e$ is the electron density, $m_e$ is the electron rest mass, $c$ is the speed of light, $e$ is the elementary charge and $\ln \Lambda$ is the Coulomb logarithm.

Runaway electrons are generated in the presence of an induced electric field parallel to the magnetic field. In a tokamak, the condition $E > E_c$ can be met during the plasma startup, during the flat-top phase of Ohmic plasmas if the density is sufficiently low, or in plasma disruptions. Especially during disruption events, a beam of runaways carrying a current of several MA and an energy of several MJ may form. Such a runaway beam would pose a serious threat to the integrity of the first wall in reactor-size fusion devices. Any mechanism that could possibly limit the formation of a considerable runaway beam would be of importance.

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While the role of radiative momentum losses due to synchrotron emission was studied previously in Andersson, Helander & Eriksson (2001), the possibility of a non-monotonic feature in the energy distribution of runaways – which we will henceforth refer to as a ‘bump’ – was not considered. The formulation of the problem in Andersson et al. (2001) does not ensure particle conservation in the presence of radiation reaction and neglects certain terms needed to describe the bump (Hazeltine & Mahajan 2004; Stahl et al. 2015). The possibility of bump formation, however, immediately raises the question of whether the non-monotonic behaviour of the distribution could lead to kinetic instabilities, causing a redistribution of the runaway particles, favourable for mitigating the potential threat to the machine. A thorough investigation of conditions favouring bump formation is thus needed.

In the present paper, we use analytical calculations to investigate the runaway electron distribution under the combined influence of Coulomb collisions, electric field acceleration and radiative momentum losses. We show the existence of a bump and derive both a threshold condition for the appearance of the bump and an approximate expression for its location in parallel momentum space. The accuracy of the analytical estimates is then tested against numerical simulations carried out using CODE (Landreman, Stahl & Fülöp 2014; Stahl et al. 2015).

The paper is organized as follows. In § 2, we start by describing the particle phase-space kinetic equation. We also discuss the transformation of the kinetic equation into the guiding-centre phase space and give the corresponding expression in the case of a uniform plasma. Analytical calculation of the condition for bump-on-tail appearance, based on the guiding-centre dynamics, is presented in § 3. A comparison of the derived conditions with numerical results is presented in § 4, before we conclude in § 5.

2. Kinetic equation

The kinetic equation describing the dynamics of charged particles in a plasma is

$$\frac{\partial f_a}{\partial t} + \frac{\partial}{\partial x} \cdot (\dot{x} f_a) + \frac{\partial}{\partial p} \cdot (\dot{p} f_a) = C[f_a, f_b],$$

(2.1)

where \(C[f_a, f_b]\) is the collision operator for collisions between particle species \(a\) and \(b\), \(z^\alpha = (x, p)\) are the phase-space coordinates and \(\dot{z}^\alpha = (\dot{x}, \dot{p})\) are the equations of motion. In the Fokker–Planck limit, the Coulomb collision operator is given by

$$C[f_a, f_b] = -\frac{\partial}{\partial p} \cdot \left( K_{ab}[f_b] f_a - D_{ab}[f_b] \cdot \frac{\partial f_a}{\partial p} \right),$$

(2.2)

where \(K_{ab}[f_b]\) is the friction vector and \(D_{ab}[f_b]\) is the diffusion tensor (see appendix A for details). In this paper, we do not consider contributions from large-angle collisions.

The equations of motion for a particle with charge \(q\) and mass \(m\) combine the Hamiltonian motion from the electric and magnetic fields \(E\) and \(B\), and a force \(F\) that accounts for non-Hamiltonian dynamics,

$$\dot{x} = v,$$

(2.3)

$$\dot{p} = qE + q v \times B + F.$$

(2.4)

Here, \(p = \gamma mv\) is the particle momentum and \(\gamma = 1/\sqrt{1 - v^2/c^2} = \sqrt{1 + p^2/(mc)^2}\) is the relativistic factor. In the case considered here, the non-Hamiltonian force is the radiation reaction (RR) force which was first described by Lorentz (1892) in the
Non-monotonic runaway tail

case of a classical non-relativistic point charge and was later generalized to relativistic
energies by Abraham (1905) and Dirac (1938). As such, the Lorentz–Abraham–Dirac
(LAD) force is (Pauli 1958)

\[ \mathbf{F}_{\text{LAD}} = \frac{q^2 \gamma^2}{6\pi \varepsilon_0 c^5} \left[ \ddot{v} + \frac{3\gamma^2}{c^2} (v \cdot \dot{v}) \dot{v} + \frac{\gamma^2}{c^2} \left( v \cdot \dddot{v} + \frac{3\gamma^2}{c^2} (v \cdot \ddot{v})^2 \right) v \right]. \]  

(2.5)

The LAD force does, however, contain third-order time derivatives of the particle
position, which allows for the existence of pathological solutions. For instance, the
particle velocity may grow exponentially in the absence of external forces \((\mathbf{E} = 0,\ \mathbf{B} = 0)\), see e.g. Rohrlich (2007). These issues have generated discussion regarding
which expression to use for the RR force. Landau & Lifshitz (1975) suggested a
perturbative approach in which the velocity derivatives in (2.5) are expressed in terms
of the external force only (here the Lorentz force). Ford & O’Connell (1993) argue
that this approach is in fact the correct one. In the paper by Spohn (2000), it is
shown that the non-physical solutions can be avoided if the LAD force is limited
on a so-called critical surface, and that the resulting expressions will be equivalent
to those of the perturbative approach. We have thus chosen to adopt the perturbative
approach. Furthermore, we neglect the electric field in the expressions for \(\dot{v}\) and \(\ddot{v}\)
in the RR force. This is justified since the motion of the particle is dominated by
the magnetic field in the strongly magnetized plasmas considered here. An excellent
discussion about the RR force can be found in a recent review paper by Di Piazza
et al. (2012).

2.1. Guiding-centre transformation

Because of the \(\mathbf{v} \times \mathbf{B}\) term, the particle phase-space kinetic equation in a magnetized
plasma includes the rapid gyro-motion time scale which is often not interesting
and is expensive to resolve computationally. It can, however, be eliminated using
guiding-centre Lie-transform perturbation methods. The transformation of the
Hamiltonian equations of motion is one of the classical results in modern plasma
physics (see Littlejohn 1983; Cary & Brizard 2009), and the Fokker–Planck collision
operator has been considered in Brizard (2004), Decker et al. (2010) and Hirvijoki
et al. (2013). The final step necessary to formulate our problem, the transformation
of the RR force, was given recently in Hirvijoki et al. (2015).

After the transformation, the guiding-centre kinetic equation for a gyro-angle
averaged distribution function \(\langle F_a \rangle\), including Hamiltonian motion in electromagnetic
fields, the RR force and Coulomb collisions in the Fokker–Planck limit, is given by

\[
\frac{\partial \langle F_a \rangle}{\partial t} + \frac{1}{\mathcal{J}} \frac{\partial}{\partial Z^a} \left[ \mathcal{J} \langle \dot{Z}^a + \langle \mathcal{F}_{\text{gcRR}}^a \rangle \rangle \langle F_a \rangle \right] = C_{\text{gcFP}}[\langle F_a \rangle],
\]

(2.6)

where the \(Z^a\) form the 5D guiding-centre phase space, \(\mathcal{J}\) is the guiding-centre
phase-space Jacobian, \(\dot{Z}^a\) are the Hamiltonian guiding-centre equations of motion
and \(\langle \mathcal{F}_{\text{gcRR}}^a \rangle \) is the contribution from the RR force to the guiding-centre motion.
Similarly to the particle phase-space operator in (2.2), we can write the guiding-centre
Fokker–Planck collision operator in phase-space divergence form,

\[
C_{\text{gcFP}}[\langle F_a \rangle] = -\frac{1}{\mathcal{J}} \frac{\partial}{\partial Z^a} \left[ \mathcal{J} \left( \langle \mathcal{K}_{ab,gc}^a \rangle \langle F_a \rangle - \langle \mathcal{D}_{ab,gc}^a \rangle \frac{\partial \langle F_a \rangle}{\partial Z^b} \right) \right],
\]

(2.7)

where \(\langle \mathcal{K}_{ab,gc}^a \rangle\) and \(\langle \mathcal{D}_{ab,gc}^a \rangle\) are the guiding-centre Coulomb friction and diffusion
coefficients.
2.2. Equations of motion

We solve (2.6) in a uniform plasma, using 2D guiding-centre velocity space coordinates \( Z^\alpha = (p, \xi) \), where \( p \) is the absolute value of the guiding-centre momentum, \( \xi = p_\parallel/p \) is the pitch angle cosine (\( p_\parallel \) is the guiding-centre momentum parallel to the magnetic field) and the guiding-centre Jacobian is given by \( J \equiv p^2 \). In this case, the guiding-centre equations of motion take the simple forms

\[
\dot{p} = qE_\parallel \xi, \quad \dot{\xi} = qE_\parallel (1 - \xi^2)/p, \tag{2.8}
\]

where \( E_\parallel \) is the electric field parallel to the magnetic field. The components of the guiding-centre RR force in the limit corresponding to pure synchrotron emission are (Hirvijoki et al. 2015)

\[
\langle F_{p gcRR} \rangle = -\gamma p (1 - \xi^2)/\tau_r, \tag{2.10}
\]
\[
\langle F_{\xi gcRR} \rangle = \xi (1 - \xi^2)/\gamma \tau_r, \tag{2.11}
\]

where the radiation reaction time scale is defined by

\[
\tau_r = \frac{6\pi\varepsilon_0 (mc)^3}{q^4 B^2} = \frac{3c}{2r\gamma^2 \Omega^2}, \tag{2.12}
\]

with \( r = q^2/(4\pi\varepsilon_0 mc^2) \) the classical electron radius and \( \Omega = qB/(\gamma m) \) the gyro-frequency.

2.3. Collision operator

The particle phase-space friction and diffusion coefficients, \( K_{ab}[f_b] \) and \( D_{ab}[f_b] \), are expressed in terms of the five relativistic Braams–Karney potential functions, which are weighted integrals of the background distribution functions \( f_b \) (see Braams & Karney 1989 and appendix A for details). If the particle species \( a \) and \( b \) coincide, the self-collisions result in a nonlinear collision operator. In the present study, the particle phase-space collision operator is transformed into the guiding-centre phase space and linearized around a Maxwellian. The integral terms of the linearized collision operator are neglected and only the test particle contribution is considered. This choice, with some further simplifications, allows analytical solution of (2.6), which will be discussed in § 3.

The guiding-centre friction and diffusion coefficients \( \langle \mathcal{K}_{ab,gc}^{\alpha} \rangle \) and \( \langle \mathcal{D}_{ab,gc}^{\alpha\beta} \rangle \) that appear in the guiding-centre Fokker–Planck operator in (2.7) are gyro-averaged projections of their guiding-centre pushed-forward particle phase-space counterparts. For a detailed definition of the guiding-centre friction and diffusion coefficients, we refer to Brizard (2004), Decker et al. (2010) and Hirvijoki et al. (2013). The general expressions are non-trivial but, in the limit of a uniform plasma, the test-particle operator assuming isotropic background particle distributions becomes diagonal with reasonably simple non-zero components

\[
\langle \mathcal{K}_{ab,gc}^{p} \rangle \equiv -v_{1,ab} p, \tag{2.13}
\]
The coefficients $\nu_{l,ab}$, $D_{l,ab}$ and $D_{t,ab}$, where the sub-indices $l$ and $t$ stand for 'longitudinal' and 'transverse' with respect to the guiding-centre momentum vector, are expressed in terms of the five Braams–Karney potentials $\Psi_n(u)$ with $u = p/m_a$ and $\Gamma_{ab} = q_a^2 q_b^2 \ln \Lambda/(4\pi\varepsilon_0^2)$ according to

$$\nu_{l,ab} = 4\pi m_a/m_b \Gamma_{ab} \gamma \left( \frac{\partial \Psi_1}{\partial u} - \frac{2}{c^2} \frac{\partial \Psi_2}{\partial u} \right),$$

$$D_{l,ab} = -4\pi \Gamma_{ab} \gamma \left( \frac{2\gamma^2}{u} \frac{\partial \Psi_3}{\partial u} + \frac{8\gamma^2}{uc^2} \frac{\partial \Psi_4}{\partial u} - \frac{8}{c^4} \Psi_4 \right),$$

$$D_{t,ab} = -4\pi \Gamma_{ab} \gamma \left( \frac{1}{u} \frac{\partial \Psi_3}{\partial u} + \frac{1}{c^2} \Psi_3 - \frac{4}{uc^2} \frac{\partial \Psi_4}{\partial u} + \frac{4}{c^4} \Psi_4 \right).$$

Our guiding-centre Fokker–Planck operator thus becomes

$$C_{gcFP}[\langle F_a \rangle] = \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 \left( \nu_{l,ab} p \langle F_a \rangle + D_{l,ab} \frac{\partial \langle F_a \rangle}{\partial p} \right) \right] + \frac{D_{t,ab}}{p^2} \frac{\partial}{\partial \xi} \left[ (1 - \xi^2) \frac{\partial \langle F_a \rangle}{\partial \xi} \right],$$

(2.19)

where the first term with momentum derivatives is responsible for the slowing down of fast particles and momentum diffusion, while the second term describes scattering in pitch angle.

2.4. Final expression

For the rest of the paper, to streamline notation, we shall suppress the brackets that denote the gyro-averaging and the sub-index from the expression for the parallel electric field. We will also drop the particle species indices as we sum over all the background species in the collision operator. Thus, we will have $\nu_l = \sum_b \nu_{l,ab}$, $D_l = \sum_b D_{l,ab}$ and $D_t = \sum_b D_{t,ab}$, and our kinetic equation in the continuity form becomes

$$\frac{\partial F}{\partial t} + \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 \left( qE \xi - \frac{\gamma p(1-\xi^2)}{\tau_r} - v_l p \right) F - p^2 D_l \frac{\partial F}{\partial p} \right]$$

$$+ \frac{\partial}{\partial \xi} \left[ (1 - \xi^2) \left( \frac{qE}{p} F + \frac{\gamma \xi}{\tau_r} F - D_t \frac{\partial F}{\partial \xi} \right) \right] = 0.$$  

(2.20)

In the following, we analyse this equation in detail. We describe the formation of a bump-on-tail in the electron distribution function both analytically and numerically. We also study the threshold conditions for the bump formation and the minimum energy of the bump location.

3. Characteristics of a bump-on-tail feature

The RR force in a straight magnetic field system increases with the square of the perpendicular momentum, $s_\perp^2 = s^2(1 - \xi^2)$. As a consequence, the extent of
the distribution function will, qualitatively, be limited in $s_{\perp}$ to a region where the parallel component of the total force acting on an electron is positive. Electrons with higher perpendicular momenta are decelerated since the radiation reaction force overcomes the acceleration due to the parallel electric field. Compared with the case without the RR force, where the distribution function is continuously expanding in $s_{\perp}$ for increasing values of the parallel momentum $s_{\parallel} = \xi s$, the limited extent of the distribution in $s_{\perp}$ when the RR force is included leads to qualitatively different dynamics.

The width of the distribution in $s_{\perp}$ is approximately constant, which means that pitch angle scattering is increasingly more effective at higher $s_{\parallel}$ in moving the electrons to the region of phase space where they are decelerated. A consequence of this is that a true steady-state solution of the kinetic equation exists and the distribution function decays exponentially in the far tail, something that was also observed in previous works, such as Andersson et al. (2001). Another new feature is the possibility of non-monotonic behaviour in the tail of the steady-state distribution function. It should be noted that this feature cannot be correctly described if the RR force is not implemented in the phase-space divergence form that conserves the phase-space density. Therefore, it is overlooked by some earlier studies. To understand the properties of the bump, and its formation, we will start the following analysis by assuming that a bump exists in the runaway tail, and make assumptions regarding its properties. These assumptions will be justified a posteriori when our results are compared with numerical results in §4. For further numerical results and insights into the dynamics of the bump, including its temporal development, we refer the reader to Decker et al. (2015).

Considering a possible bump-on-tail scenario, we study (2.20) in a region where the electrons have high velocities compared with the electron and ion thermal speeds, $v \gg v_{th,e}, v_{th,i}$. Then, the slowing-down force is dominated by electron–electron collisions and it overshadows momentum diffusion. For pitch angle scattering, collisions with both ions and the electron bulk are important.

In the limit where the bulk populations are non-relativistic Maxwellians, we have for the friction coefficient at high speeds

$$v_l \approx \frac{n_e e^4 \ln \Lambda}{4\pi\varepsilon_0^2 m_e v^2 p} \equiv \frac{eE_c}{\beta^2 p},$$

(3.1)

and similarly for the transverse diffusion coefficient

$$D_t \approx \frac{1 + Z_{eff} e^4 \ln \Lambda}{2 \times \frac{n_e e^4}{4\pi\varepsilon_0^2} v} = \frac{1 + Z_{eff} eE_c m_e c}{\beta},$$

(3.2)

where $E_c$ is the critical electric field (1.1), $Z_{eff}$ is the effective ion charge and $\beta = v/c$. These estimates coincide with the expressions in Andersson et al. (2001). We define the normalized momentum $s = p/(m_e c)$, time $\tau = eE_{\parallel} t/(m_e c)$, radiation reaction time scale $\sigma^{-1} = eE_c \tau_e/(m_e c)$ and electric field $\hat{E} = -E/E_c$, and transform the kinetic equation into a dimensionless form for further analysis,

$$\frac{\partial F}{\partial \tau} + \frac{1}{s^2} \frac{\partial}{\partial \xi} \left[ s^2 \left( \hat{E} \xi - \sigma \gamma s (1 - \xi^2) - \frac{\gamma^2}{s^2} \right) F \right]$$

$$+ \frac{\partial}{\partial \xi} \left[ (1 - \xi^2) \left( \frac{\hat{E} F}{s} + \frac{\sigma \xi}{\gamma} F - \frac{\gamma}{s} \frac{1 + Z_{eff} \sigma \gamma}{2s^2} \frac{\partial F}{\partial \xi} \right) \right] = 0.$$  

(3.3)
As the electric field affects only the parallel acceleration we expect the system to be strongly biased about $\xi = 1$. The phase-space volume element in $(s, \xi)$ coordinates ($\mathcal{J} \sim s^2$), however, scales nonlinearly close to $\xi \approx 1$. A better choice for further studies close to the $\xi = 1$ region is to use coordinates $(s_\parallel, s_\perp)$ that have a Jacobian $\mathcal{J} \sim s_\perp$ that stays constant with respect to $s_\parallel$. The new coordinates relate to $(s, \xi)$ according to

$$s_\parallel = s\xi,$$

$$s_\perp = s\sqrt{1 - \xi^2},$$

and our kinetic equation expressed with $(s_\parallel, s_\perp)$ becomes

$$\frac{\partial F}{\partial \tau} + \hat{E} \frac{\partial F}{\partial s_\parallel} - 2F - \frac{\gamma^2}{s^2} \left( \frac{s_\parallel \partial F}{s \partial s_\parallel} + \frac{s_\perp \partial F}{s \partial s_\perp} \right) = -\frac{\gamma}{2s} \left[ \frac{\partial^2 F}{\partial s_\parallel \partial s_\perp} + \frac{s_\perp^2}{s^2} \left( \frac{\partial^2 F}{\partial s_\parallel^2} - \frac{\partial^2 F}{\partial s_\perp^2} \right) \right] - 2\frac{s_\parallel s_\perp}{s^2} \frac{\partial^2 F}{\partial s_\parallel \partial s_\perp} - 2\frac{s_\parallel}{s^2} \left( \frac{\partial F}{\partial s_\parallel} + \frac{s_\perp}{s_\parallel} \frac{\partial F}{\partial s_\perp} \right)$$

$$- \frac{\sigma}{\gamma} \left( 2 + 4s_\perp^2 \right) F + s_\parallel \left( 1 + s_\perp^2 \right) \frac{\partial F}{\partial s_\parallel} + s_\parallel s_\perp^2 \frac{\partial F}{\partial s_\perp} = 0. \quad (3.6)$$

Instead of attempting to solve (3.6), in the following we will concentrate on the dynamics at $s_\perp = 0$, which will be sufficient to prove the existence of a bump and to estimate its location in the electron tail.

We assume the distribution to be a smooth function of $s_\perp$, which allows us to create a power series expansion around $s_\perp = 0$:

$$F(s_\parallel, s_\perp) = \sum_{n=0}^{\infty} \frac{s_\perp^{2n}}{(2n)!} \left[ \frac{\partial^{(2n)} F}{\partial s_\perp^{(2n)}} \right]_{(s_\parallel, 0)} + s_\parallel \sum_{n=0}^{\infty} \frac{s_\perp^{2n}}{(2n + 1)!} \left[ \frac{\partial^{(2n+1)} F}{\partial s_\perp^{(2n+1)}} \right]_{(s_\parallel, 0)}. \quad (3.7)$$

Because the electric field is acting only in the parallel direction, $F$ is ‘even’ in $s_\perp$, i.e. we can formally state that $F(s_\parallel, s_\perp) = F(s_\parallel, -s_\perp)$ although our phase space does not extend to $s_\perp < 0$. Thus, all the odd $s_\perp$ derivatives at $s_\perp = 0$ must vanish, and we find

$$F(s_\parallel, s_\perp) = \sum_{n=0}^{\infty} \frac{s_\perp^{2n}}{(2n)!} \left[ \frac{\partial^{(2n)} F}{\partial s_\perp^{(2n)}} \right]_{(s_\parallel, 0)}. \quad (3.8)$$

With the help of the expansion, we may accurately calculate the limit

$$\lim_{s_\perp \to 0} \frac{1}{s_\perp} \frac{\partial}{\partial s_\perp} \left( s_\perp \frac{\partial F}{\partial s_\perp} \right) = 2 \left[ \frac{\partial^2 F}{\partial s_\perp^2} \right]_{(s_\parallel, 0)}. \quad (3.9)$$
and write the $s_{\perp} = 0$ limit of the kinetic equation as
\[
\left[ \frac{\partial F}{\partial \tau} \right]_{(s_{\perp}, 0)} + \left( \hat{E} - \frac{1 + s_{\perp}^2}{s_{\parallel}^2} + \frac{(1 + Z_{\text{eff}})\sqrt{1 + s_{\parallel}^2}}{s_{\parallel}^2} \right) \left[ \frac{\partial F}{\partial s_{\parallel}} \right]_{(s_{\perp}, 0)} - \frac{(1 + Z_{\text{eff}})\sqrt{1 + s_{\parallel}^2}}{s_{\parallel}} \left[ \frac{\partial^2 F}{\partial s_{\parallel}^2} \right]_{(s_{\perp}, 0)} - 2 \left( \frac{\sigma}{\sqrt{1 + s_{\parallel}^2}} + \frac{1}{s_{\parallel}} \right) [F]_{(s_{\perp}, 0)} = 0. \tag{3.10}
\]

Assuming that a steady-state solution exists, the possible extrema are characterized by the condition
\[
\left[ \frac{\partial F}{\partial s_{\parallel}} \right]_{(s_{\perp}, 0)} = 0. \tag{3.11}
\]

We thus find an algebraic equation that defines the locations of these extrema:
\[
2 \left( \frac{\sigma}{\sqrt{1 + s_{\parallel}^2}} + \frac{1}{s_{\parallel}} \right) + \frac{(1 + Z_{\text{eff}})\sqrt{1 + s_{\parallel}^2}}{s_{\parallel}} \left[ \frac{\partial^2 F}{\partial s_{\parallel}^2} \right]_{(s_{\perp}, 0)} = 0. \tag{3.12}
\]

### 3.1. Threshold condition for the appearance of the bump

Considering a steady-state solution to (3.10), in a situation where the bump is on the verge of appearing, a single inflection point exists in the distribution function instead of local maxima or minima. In this section we derive a threshold condition describing the appearance of an inflection point, by requiring the first and second $s_{\parallel}$ derivatives of the distribution to vanish simultaneously.

Before we start the analysis, we note that the steady-state distribution function represented by equation (12) of Andersson et al. (2001) is separable in $s_{\parallel}$ and $s_{\perp}$, and it is of the form $\alpha \exp[-W_{\infty}^2 s_{\perp}^2/2]$, where $W_{\infty}^2 = 2\sigma/(\hat{E} - 1)$. To find this result they neglect $f/s_{\parallel}$ corrections compared with $\partial f/\partial s_{\parallel}$ terms in the kinetic equation, which is appropriate in the very far tail ($s_{\parallel}$ corresponds to $p_{\parallel}$ in the notation of Andersson et al. 2001). Therefore, the quantity
\[
W^2(s_{\parallel}) \equiv - \left[ \frac{1}{F} \frac{\partial^2 F}{\partial s_{\perp}^2} \right]_{(s_{\parallel}, 0)} \tag{3.13}
\]

should approach $W_{\infty}^2$ in the $s_{\parallel} \to \infty$ limit. Thus, it is useful to define $\kappa(s_{\parallel})$, so that $W^2(s_{\parallel}) = \kappa(s_{\parallel}) W_{\infty}^2$ and $\kappa \to 1$ as $s_{\parallel} \to \infty$. Numerical calculations tell us that $0 < \kappa \leq 1$ for the regions of interest in the runaway tail, and $\kappa$ is often a slowly varying function of $s_{\parallel}$. That is, the characteristic width of the distribution function in the $s_{\perp}$ direction, $1/W^2$, decreases with increasing $s_{\parallel}$, and slowly asymptotes to a constant value. For now, we may simply use $0 < \kappa \leq 1$ as a working hypothesis to be verified through numerical calculations later.

We start with (3.12) satisfied at extrema or inflection of the distribution function and rewrite it as
\[
L(s_{\parallel}) \equiv 2 \left( \sigma s_{\parallel} + \sqrt{1 + s_{\parallel}^2} \right) - K(s_{\parallel})(1 + s_{\perp}^2) = 0, \tag{3.14}
\]
where \( K(s_\parallel) = W_\infty^2 (1 + Z_{\text{eff}}) \kappa(s_\parallel) = \bar{E}^{-1} \sigma \kappa(s_\parallel) \) with \( \bar{E} = (\hat{E} - 1)/[2(1 + Z_{\text{eff}})] \). It is useful to form

\[
L'(s_\parallel)/2 \approx \sigma + \frac{s_\parallel}{\sqrt{1 + s_\parallel^2}} - K(s_\parallel)s_\parallel,
\]

where the prime denotes a derivative with respect to \( s_\parallel \), and we neglected a term, \(-K'(s_\parallel)(1 + s_\parallel^2)/2\), assuming \( \kappa \) to be a sufficiently slowly varying function of \( s_\parallel \). From (3.15) we see that \( L'(s_\parallel) \to -\infty \) as \( s_\parallel \to \infty \), while \( L'|_{s_\parallel=0} = 2\sigma > 0 \). It can also be shown that \( L' = 0 \) only has one root for positive values of \( s_\parallel \), which then has to correspond to a single maximum of \( L \).

If the distribution has an inflection point, both \( L \) and \( L' \) should vanish there. Assuming the slowly varying \( \kappa \) to be a constant, the system of (3.14) and (3.15) can be solved for \( s_\parallel \) and \( K \) to find

\[
K_0 \equiv K(s_{\parallel|0}) = (4\sqrt{2}\sigma)^{-1} \left[ 8\sigma \sqrt{4 - \frac{8}{3 + \sqrt{1 + 8\sigma^2}}} + \left( 1 + 4\sigma^2 - \sqrt{1 + 8\sigma^2} \right) \sqrt{2 - \frac{8}{3 + \sqrt{1 + 8\sigma^2}}} \right],
\]

and

\[
s_{\parallel|0} = \sqrt{1 - \frac{4}{3 + \sqrt{1 + 8\sigma^2}}},
\]

where the subscript 0 refers to values of quantities at the threshold of the bump appearance. By inspecting the expressions (3.16) and (3.17) we find that both of them increase with \( \sigma \) monotonically; \( K_0 \) between 2 and \( \infty \), and \( s_{\parallel|0} \) between 0 and 1. This means that an inflection point is always located below \( s_\parallel = 1 \). Note, that we assume the inflection point to be sufficiently far from the bulk, and that \( \kappa' \) is small; violation of these assumptions may move \( s_{\parallel|0} \) above unity. However, since this problem cannot be addressed until the more complete (2.20) is solved, we assume that the conditions above are fulfilled in order to proceed analytically. Since \( s_\parallel < 1 \), we can make use of the expansion

\[
\sqrt{1 + s_\parallel^2} = 1 + \frac{s_\parallel^2}{2} + O(s_\parallel^4).
\]

By neglecting \( O(s_\parallel^4) \) terms (which is a reasonably good approximation even when \( s_\parallel \) approaches unity), (3.14) becomes quadratic in \( s_\parallel \):

\[
2(\sigma s_\parallel + 1 + s_\parallel^2/2) - K(s_\parallel)(1 + s_\parallel^2) = 0.
\]

At the inflection point \( s_{\parallel|0} \), (3.19) must have a single root, which requires the discriminant to vanish. This determines the threshold value of \( K \):

\[
K_0 = \left( 3 + \sqrt{1 + 4\sigma^2} \right) / 2,
\]

which is positive. This can be substituted back into (3.19) to find

\[
s_{\parallel|0} = \sigma/(K_0 - 1) = \sigma \left[ \left( 1 + \sqrt{1 + 4\sigma^2} \right) / 2 \right]^{-1}.
\]
Combination of \( K(s_\parallel) = \tilde{E}^{-1} \sigma \kappa(s_\parallel) \) with (3.20) to solve for a positive \( \sigma \) that corresponds to \( K \geq 2 \) yields \( \sigma \) as a function of \( \tilde{E} \) at the threshold for bump formation:

\[
\sigma_0 = \frac{3\kappa/\tilde{E} + \sqrt{8 + \kappa^2/\tilde{E}^2}}{2(\kappa^2/\tilde{E}^2 - 1)},
\]

where \( \kappa = \kappa(s_\parallel = s_\parallel^0) \leq 1 \) is treated as a parameter. When \( \sigma \) is increased above \( \sigma_0 \), \( L \) becomes negative and no bump appears. Reducing \( \kappa \) below unity increases the threshold value of \( \sigma \). Thus, (3.22) with \( \kappa = 1 \) represents an absolute lower threshold in \( \sigma \) for a monotonic behaviour of the steady-state distribution function. Furthermore, even at \( \kappa = 1 \) the threshold is limited to the region \( \tilde{E} < 1 \); thus a bump should always appear when \( \tilde{E} \geq 1 \).

We have considered \( K(s_\parallel = 0) > 2 \), in which case \( L(s_\parallel) = 0 \) can have zero (no bump), one (threshold) or two (bump exists) positive real roots. When \( K(s_\parallel = 0) < 2 \) there is only a single positive root of \( L(s_\parallel) \). Since the distribution function cannot have a positive slope in the high-\( s_\parallel \) limit this root should also correspond to a bump. That, however, requires the existence of a minimum in the distribution function along the positive \( s_\parallel \) axis, which must then appear outside the domain of validity of (3.12). In fact, this minimum will appear close to the bulk part of the electron distribution, where neglected corrections to the collision operator become important.

We can conclude that for \( \tilde{E} \geq 1 \), there should always be a bump in the steady-state distribution function as long as there is a finite magnetic field. This, perhaps somewhat counterintuituitive, result needs some clarification to accommodate the well-known \( \sigma = 0 \) limit. When no loss mechanisms are considered (in this case when \( \sigma = 0 \)), the electron distribution has no steady-state solution, and the runaway tail at \( s_\perp = 0 \) should converge to a \( 1/s_\parallel \) decay. When \( \sigma \) is small, the bump location moves to high values in \( s_\perp \), as will be shown in the next section. The runaway tail always builds up starting from the bulk, and, for a tiny \( \sigma \), the process may take such a long time that the distribution never becomes non-monotonic in practice. In this scenario the steady-state distribution and the bump have no relevance. Moreover, other loss mechanisms may limit the distribution function to momenta below \( s_\parallel^0 \) in realistic cases.

### 3.2. An estimate for the bump location in the far tail

In order to proceed and estimate the location of the bump and the shape of the distribution function, we look for a steady-state solution in a region where the guiding-centre parallel momentum is large. Using the expansion

\[
\sqrt{1 + s_\parallel^2} = s_\parallel + O(s_\parallel^{-1}),
\]

(3.10) gives

\[
(\hat{E} - 1 + O(s_\parallel^{-1})) \left[ \frac{\partial F}{\partial s_\parallel} \right]_{(s_\parallel,0)} - (1 + Z_{\text{eff}}) \left[ \frac{\partial^2 F}{\partial s_\perp^2} \right]_{(s_\parallel,0)} - 2 \frac{1 + \sigma}{s_\parallel} [F]_{(s_\parallel,0)} = 0.
\]

We neglect the \( O(s_\parallel^{-1}) \) term, which is valid if \( \hat{E} - 1 \) is not very small, and assume that the width of the distribution function in the \( s_\perp \) direction, and thus \( W^2 = -F^{-1}(\partial^2 F/\partial s_\perp^2)|_{(s_\parallel,0)} \), stays approximately constant close to the bump. This essentially
means that we are looking for a separable solution of the form \( F \sim h(s_{\parallel})g(s_{\perp}) \). We obtain an ordinary differential equation

\[
(\hat{E} - 1)s_{\parallel}h' - 2(1 + \sigma)h = -(1 + Z_{\text{eff}})W^2s_{\parallel}h,
\]

which is solved by

\[
h(s_{\parallel}) \sim s_{\parallel}^{2(1+\sigma)/(\hat{E}-1)} \exp\left[-W^2(1 + Z_{\text{eff}})/(\hat{E} - 1)s_{\parallel}\right],
\]

and the location of the bump-on-tail is given by

\[
s_{\parallel} = \frac{1 + \sigma}{1 + Z_{\text{eff}}} \frac{2}{W^2_{\infty} \kappa}.
\]

If we again assume that \( \kappa \) does not exceed unity, recalling \( W^2_{\infty} = 2\sigma/(\hat{E} - 1) \), we find a lower bound for the parallel momentum at the bump,

\[
s_{\parallel,\text{min}} = \frac{1 + \sigma}{\sigma} \frac{\hat{E} - 1}{1 + Z_{\text{eff}}} = \frac{1 + \sigma}{\sigma} 2\bar{E}.
\]

We see that for small values of \( \sigma \), the bump would appear at high parallel momenta. By setting \( s_{\parallel,\text{min}} \) to some upper limit of physical interest, \( s_{\parallel,\text{L}} \), (3.28) may be used to find an estimate for a lower ‘practical limit’ in \( \sigma \) for the appearance of the bump. Namely, if \( \sigma \) is smaller than

\[
\sigma_{\text{L}} = \frac{1}{(s_{\parallel,\text{L}} \kappa)/(2\bar{E}) - 1},
\]

for \( \kappa = 1 \), then a bump would only appear at some large parallel momentum above \( s_{\parallel,\text{L}} \), which is then deemed physically irrelevant. It should be noted that if the bump is in the far tail, \( \kappa \) can be significantly less than unity, as will be shown in the next section, using numerical simulations. Letting \( \kappa < 1 \) increases the practical limit in \( \sigma \). Another implication of (3.29) is that for a normalized electric field higher than \( \bar{E} = s_{\parallel,\text{L}}/2 \), the bump always appears above \( s_{\parallel,\text{L}} \) for any value of \( \sigma \).

4. Comparison with numerical results

The numerical results shown in this section were performed with the continuum simulation tool CODE used in its time-independent mode. CODE solves the two-dimensional momentum space kinetic equation in a homogeneous plasma, using a linearized Fokker–Planck operator valid for arbitrary electron energies. For a detailed description of the tool, see Landreman et al. (2014).

First, we provide a typical example of a non-monotonic runaway distribution function in the presence of radiation reaction. Figure 1(a) shows the momentum dependence of the pitch angle averaged runaway electron distribution with \( (B = 2.5 \text{ T}) \) and without \( (B = 0 \text{ T}) \) radiation reaction force, plotted with solid and dashed curves respectively. Technically, the pitch angle averaged distribution is the lowest mode in a Legendre polynomial expansion of \( F \) in \( \xi \), normalized so that \( F \) is unity at its maximum. The simulations were performed with the parameters \( T_e = 5 \text{ keV} \), \( n_e = 2 \times 10^{19} \text{ m}^{-3} \), \( Z_{\text{eff}} = 1.2 \) and \( \hat{E} = 2 \). It should be noted that the distribution
Figure 1. Typical examples of non-monotonic runaway electron distribution functions. (a) The pitch angle average of the distribution function with (solid curve) and without (dashed) synchrotron radiation reaction. A Maxwellian distribution is also indicated (dash-dotted). (b) Contour plot of the distribution function corresponding to the solid curve in (a), as a function of \( s_\parallel \) and \( s_\perp \).

function without radiation reaction represents a quasi-steady state. The lack of loss mechanisms leads to a slow but steady depletion of the bulk electron population, as more and more electrons run away and leave the computational domain. This outflow must be balanced by an artificial source of thermal (Maxwellian) electrons to maintain the quasi-steady state. In the presence of radiation reaction, the distribution is a true steady state. When the radiation reaction is included, the non-monotonic feature is present when the distribution is averaged over pitch angles in the present example. However, we note that for less pronounced bumps, the pitch angle averaged distribution can have a monotonic tail, or it may exhibit a bump at some values of \( s \) that are appreciably lower than those where the bump is observed in the full 2D distribution. This may have an impact on the possibility for bump-on-tail-type instabilities to arise. Figure 1(b) shows a contour plot in \( s_\parallel \)–\( s_\perp \) momentum space of the distribution function corresponding to the solid curve in figure 1(a). Although this example is representative of a typical runaway electron distribution, the location and height of the bump, and the width of the distribution in \( s_\perp \) can vary significantly depending on the plasma parameters. The relation between the location \( s_\parallel \) and the local ‘width’ (\( \sim 1/W^2 = 1/(W_{\infty, k}^2) \)) of the distribution given by (3.27) is accurate as long as the location of the bump is not close to unity, i.e. sufficiently far from the no-bump threshold (3.22). This justifies the approximations applied to the collision operator in our analysis. In particular, energy diffusion can be neglected since no sharp features of the distribution function in \( s_\parallel \) are present, as seen in figure 1(b).

In order to investigate the validity of our analytical calculations, we have performed a numerical analysis of the appearance of the bump by scanning the parameter space
with CODE. The electron temperature and density were held constant at the values \( T_e = 1 \text{ keV} \) and \( n_e = 5 \times 10^{18} \text{ m}^{-3} \) respectively, while the magnetic field, the induced electric field and the effective ion charge were varied over the ranges \( B \in [1, 6] \text{ T} \), \( \hat{E} \in [2, 14] \) and \( Z_{\text{eff}} \in [1, 3] \). The numerical calculations used 950 momentum grid points, 130 Legendre modes for the decomposition in \( \xi \), and a highest resolved momentum of \( s = 34 \), providing well-converged solutions.

The results of the scan are presented in figure 2, where circles and crosses correspond to distributions with and without a bump respectively. The colour coding of the circles reflects the location of the bump, with 100% in the colour bar corresponding to \( s_\parallel = 34 \). Simulations with a bump appearing above 80% (\( s_\parallel = 27 \)) are excluded from the figure, since those results may be affected by the bump being too close to the highest resolved momentum. As expected from (3.28), increasing \( \bar{E} \) or decreasing \( \sigma \) moves the bump towards larger momenta.

A reasonably good agreement is found between the numerical calculations and the analytical threshold for the bump to exist. The solid curve shows this threshold, (3.22), for \( \kappa = 1 \). Above \( \sigma \approx 0.5 \) the ‘no bump’ solutions obey the analytical threshold and fall to the left of the threshold curve. There are some solutions with a bump in this region as well. However, this is not surprising, since \( \kappa \) at the bump is allowed to be less than unity, in which case the threshold moves towards lower values of \( \bar{E} \). The threshold begins to fail for lower values of \( \sigma \), showing that the approximation \( K' \ll \{\sigma, s_\parallel\} \) used in (3.15) breaks down. Nevertheless, the qualitative behaviour of the threshold is still captured by (3.22). The lower right corner of the plot (high \( \bar{E} \) and small \( \sigma \)) is not populated, since some simulations where the bump would have appeared at too high \( s_\parallel \) were excluded. With \( s_{\parallel L} = 27 \), a \( \kappa \) value as low as 0.3 is needed in order for the limit given by (3.29) (dashed line) to correspond well to the boundary of the region of excluded points. Thus, \( \kappa \) can be significantly lower than unity at a bump with large momentum.

From the parameter scan used to generate figure 2, the simulations exhibiting bumps were compared with the theoretical lower bound for the location of the bump (3.28). As shown in figure 3, we find that, indeed, the parallel momenta at the bumps (shown


Figure 3. Parallel momentum of the bump. Circles denote the locations of the bumps according to numerical solutions, while the solid line represents a theoretical lower limit, (3.28). In the simulations, all bumps appear above the analytical threshold condition.

with green circles) are all higher than the lower bound (solid line). In fact, most of the $s_{\parallel}$ values are well above this limit. This merely confirms our finding that $\kappa$ at the bump is typically less than unity, especially for parameters sufficiently exceeding the no-bump threshold.

5. Conclusions

We have analysed the runaway electron distribution function, accounting for the radiation reaction force, and have shown that the steady-state runaway distribution can become non-monotonic. Furthermore, a threshold condition for the appearance of the bump, as well as a lower limit to its location in momentum space, was derived. While slowing down and pitch angle scattering due to Coulomb collisions are taken into account in our analysis, we do not consider the effect of large-angle collisions, and we restrict the study to a straight magnetic field geometry. Our analytical results show good agreement with numerical simulations obtained using the CODE solver.

We find that for a normalized electric field larger than unity, $\bar{E} > 1$, the steady-state electron distribution always exhibits a bump, independently of the value of the $\sigma$ parameter quantifying the strength of the radiation reaction, as long as the magnetic field is non-vanishing ($\sigma > 0$). For a smaller electric field, the appearance of the bump is well correlated with the $\sigma$ threshold given by (3.22). Although above this threshold there must always be a bump in the steady-state distribution function, it may not have a practical relevance in some cases. When $\sigma$ is small and/or $\bar{E}$ is large, the bump would be located at a very large parallel momentum, but the forefront of the electron distribution can require a long time to reach that far. This motivates the introduction of another ‘practical’ threshold condition, (3.29). If $\sigma$ is lower than this threshold, the bump will appear at a momentum above some specified limit, $s_{\parallel, L}$, and can then be considered unimportant. In particular, above a normalized electric field of $\bar{E} = s_{\parallel, L}/2$, this criterion is satisfied for any $\sigma$.

Nevertheless, when the radiation reaction is strong enough and/or the parallel electric field is not too high, there is a possibility for a bump to form in the
runaway tail. This non-monotonic feature presents a potential source of bump-on-tail instabilities, which can play a role in limiting the formation of large runaway beams.

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Appendix A. The relativistic collision operator
The relativistic particle phase-space Beliaev–Budker collision operator in the Landau form is defined as

$$ C[f_a, f_b] = -\frac{\Gamma_{ab}}{2} \frac{\partial}{\partial p} \cdot \int \frac{d\mathbf{p}'}{2\pi^3} \mathbf{U}(\mathbf{u}, \mathbf{u}') \cdot \left( f_a \frac{\partial f_b}{\partial \mathbf{p}'} - f_b \frac{\partial f_a}{\partial \mathbf{p}} \right), \quad (A\ 1) $$

where $$ \Gamma_{ab} = e_a e_b^2 \ln \Lambda/(4\pi\epsilon_0^2), \quad \mathbf{u} = \mathbf{p}/m_a, \quad \mathbf{u}' = \mathbf{p'}/m_b, $$ and the collision kernel is given by

$$ \mathbf{U}(\mathbf{u}, \mathbf{u}') = \frac{r^2}{\gamma\gamma'w^3} [w^2 \mathbf{l} -uu -u'u' + r(uu' + u'u)], \quad (A\ 2) $$

with the coefficients

$$ \gamma = \sqrt{1 + (u/c)^2}, \quad (A\ 3) $$
$$ \gamma' = \sqrt{1 + (u'/c)^2}, \quad (A\ 4) $$
$$ r = \gamma \gamma' - \mathbf{u} \cdot \mathbf{u}'/c^2, \quad (A\ 5) $$
$$ w = c\sqrt{r^2 - 1}. \quad (A\ 6) $$

Braams & Karney (1989) found a corresponding differential form for the collision operator,

$$ C[f_a, f_b] = -\frac{\partial}{\partial \mathbf{p}} \cdot \left( \mathbf{K}_{ab}[f_b] f_a - \mathbf{D}_{ab}[f_b] \cdot \frac{\partial f_a}{\partial \mathbf{p}} \right), \quad (A\ 7) $$

where $$ \mathbf{K}_{ab}[f_b] $$ is the friction vector and $$ \mathbf{D}_{ab}[f_b] $$ is the diffusion tensor which are defined with differential operations on Braams–Karney potentials $$ \Psi_n(\mathbf{u}) $$ according to

$$ \mathbf{K}_{ab}[f_b] = -4\pi \frac{m_a}{m_b} \frac{\Gamma_{ab}}{\gamma} K \left( \Psi_1 - 2\frac{\Psi_2}{c^2} \right), \quad (A\ 8) $$
$$ \mathbf{D}_{ab}[f_b] = -4\pi \frac{\Gamma_{ab}}{\gamma} \left[ L \left( \Psi_3 - 4\frac{\Psi_4}{c^2} \right) + \frac{1}{c^2} \left( \mathbf{l} + \frac{uu}{c^2} \right) \left( \Psi_3 + 4\frac{\Psi_4}{c^2} \right) \right]. \quad (A\ 9) $$

The differential operators $$ K $$ and $$ L $$ are defined as

$$ K\Psi(\mathbf{u}) = \left( \mathbf{l} + \frac{uu}{c^2} \right) \cdot \frac{\partial \Psi}{\partial \mathbf{u}}, \quad (A\ 10) $$
$$ L\Psi(\mathbf{u}) = \left( \mathbf{l} + \frac{uu}{c^2} \right) \cdot \frac{\partial^2 \Psi}{\partial \mathbf{u} \partial \mathbf{u}} \cdot \left( \mathbf{l} + \frac{uu}{c^2} \right) + \frac{1}{c^2} \left( \mathbf{l} + \frac{uu}{c^2} \right) \left( \mathbf{u} \cdot \frac{\partial \Psi}{\partial \mathbf{u}} \right). \quad (A\ 11) $$
and the potential functions are given by the integrals

\[ \Psi_0(u) = -\frac{1}{4\pi} \int du' \frac{f_b(u')}{\gamma' w}, \]  
\[(A 12)\]

\[ \Psi_1(u) = -\frac{1}{4\pi} \int du' \frac{r f_b(u')}{\gamma' w}, \]  
\[(A 13)\]

\[ \Psi_2(u) = -\frac{1}{8\pi} \int du' \frac{\sinh^{-1}(w/c) c f_b(u')}{\gamma'}, \]  
\[(A 14)\]

\[ \Psi_3(u) = -\frac{1}{8\pi} \int du' \frac{w f_b(u')}{\gamma'}, \]  
\[(A 15)\]

\[ \Psi_4(u) = -\frac{1}{32\pi} \int du' \frac{c^3}{\gamma} (r \sinh^{-1}(w/c) - (w/c)) f_b(u'). \]  
\[(A 16)\]

Furthermore, the potential functions satisfy the differential relations

\[ L_0 \Psi_0 = f_b, \]  
\[(A 17)\]

\[ L_1 \Psi_1 = f_b, \]  
\[(A 18)\]

\[ L_1 \Psi_2 = \Psi_1, \]  
\[(A 19)\]

\[ L_2 \Psi_3 = \Psi_0, \]  
\[(A 20)\]

\[ L_2 \Psi_4 = \Psi_3, \]  
\[(A 21)\]

where the operator \( L_k \) is defined as

\[ L_k \Psi = \left( I + \frac{u u}{c^2} \right) : \frac{\partial^2 \Psi}{\partial u \partial u} + \frac{3u}{c^2} \frac{\partial \Psi}{\partial u} + \frac{1 - k^2}{c^2} \Psi. \]  
\[(A 22)\]

In the case of isotropic background distributions \( f_b(u) \), the potentials become functions of \( u \) only, and the friction vector and diffusion tensor can be simplified into

\[ K_{ab} = -4\pi \frac{m_a}{m_b} \Gamma_{ab} \gamma \left( \frac{\partial \Psi_1}{\partial u} - \frac{2 \partial \Psi_3}{u} \right) \frac{p}{p} \equiv -\nu_{l,ab} p, \]  
\[(A 23)\]

\[ D_{ab} = -4\pi \Gamma_{ab} \gamma \left( \Psi_0 - \frac{2\gamma^2}{u} \frac{\partial \Psi_3}{\partial u} + \frac{8\gamma^2}{uc^2} \frac{\partial \Psi_4}{\partial u} - \frac{8}{c^4} \Psi_4 \right) \frac{pp}{p^2} \]  
\[ - 4\pi \Gamma_{ab} \gamma \left( \frac{1}{u} \frac{\partial \Psi_3}{\partial u} + \frac{1}{c^2} \Psi_3 - \frac{4}{uc^2} \frac{\partial \Psi_4}{\partial u} + \frac{4}{c^4} \Psi_4 \right) \left( I - \frac{pp}{p^2} \right) \]  
\[ \equiv D_{l,ab} \frac{pp}{p^2} + D_{t,ab} \left( I - \frac{pp}{p^2} \right). \]  
\[(A 24)\]

The guiding-centre transformation of the Fokker–Planck operator presented in Brizard (2004) gave explicit expressions for the guiding-centre friction and diffusion coefficients, \( \langle \mathcal{X}_{ab,gc} \rangle \) and \( \langle D_{ab,gc} \rangle \), in the case of isotropic background distributions and a non-relativistic collision kernel. Generalization of that work to a relativistic collision kernel is straightforward in the case of isotropic field–particle distributions because the forms of the particle phase-space friction and diffusion coefficients do not change. Only the expressions for \( \nu_{l,ab} \), \( D_{l,ab} \), and \( D_{t,ab} \) are different, but that will not affect the guiding-centre transformation, as they are functions only of the guiding-centre kinetic momentum.
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Paper D
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Numerical calculation of ion runaway distributions

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Ions accelerated by electric fields (so-called runaway ions) in plasmas may explain observations in solar flares and fusion experiments; however, limitations of previous analytic work have prevented definite conclusions. In this work, we describe a numerical solver of the 2D non-relativistic linearized Fokker-Planck equation for ions. It solves the initial value problem in velocity space with a spectral-Eulerian discretization scheme, allowing arbitrary plasma composition and time-varying electric fields and background plasma parameters. The numerical ion distribution function is then used to consider the conditions for runaway ion acceleration in solar flares and tokamak plasmas. Typical time scales and electric fields required for ion acceleration are determined for various plasma compositions, ion species, and temperatures, and the potential for excitation of toroidal Alfvén eigenmodes during tokamak disruptions is considered. © 2015 AIP Publishing LLC.

I. INTRODUCTION

The phenomenon of particle runaway in a plasma is well known, occurring in both space and laboratory plasmas.1,2 It arises because the friction force experienced by a charged particle decreases with particle energy, so that the presence of a sufficiently strong induced electric field can allow the particle to be accelerated—or run away—to high energy.

Electron runaway has been extensively studied in the context of magnetic confinement fusion in tokamaks, where it can lead to the formation of localized high-energy beams which must be carefully controlled.3 The standard analytic method used to determine the initial evolution of the electron distribution function in a fully ionized plasma was introduced by Kruskal and Bernstein4 and later generalized by Connor and Hastie.5 It takes the form of an asymptotic expansion of the electron kinetic equation in the electric field strength. Once a fast electron population—known as the primary distribution—is established, it can rapidly produce further fast electrons through large-angle, or knock-on, collisions.6 This avalanche process leads to the so-called secondary runaway-electron generation, which is often dominant.

Ion runaway has long been of interest in the astrophysical community, where it is thought to contribute to the observed abundance of high energy ions in solar flares.3 It has also been used to study the behavior of lightning channels8 and was observed in laboratory plasmas, e.g., in the Mega Ampere Spherical Tokamak (MAST)9 and in the Madison Symmetric Torus.10 The detailed mechanism of ion runaway differs from that of electron runaway. Friction with the electrons, which are also drifting in the electric field, acts to cancel a portion of the accelerating field. In the ion rest frame, the electrons have net motion anti-parallel to the electric field, and a test ion will experience two main forces beyond friction against the background of charged particles: acceleration in the electric field and friction due to the electron drift. These forces scale differently with ion charge, and the dominant force is either electron friction—with the consequence that the ions are dragged along with the electrons—or acceleration by the electric field. In a pure plasma, the cancellation of electric field acceleration and electron friction is complete, but the presence of impurities, neutrals, or effects such as particle trapping in a non-uniform confining magnetic field allow a finite effective field to remain.11,12

Furth and Rutherford12 generalized earlier treatments by adopting a similar expansion procedure to that used to study electron runaway. They determined the steady state ion distribution function in conditions typical of operational fusion plasmas. Helander et al.9 then considered the initial value problem resulting from the onset of an accelerating electric field, produced, for example, by a plasma instability. An analytic solution for the initial time evolution of the accelerated ion distribution function was developed, but it was noted that its application was limited due to the low electric fields required for it to be valid. Both of these ion runaway studies considered only the presence of a trace impurity population, consistent with typical operating conditions in fusion plasmas.

Plasmas with significant impurity content are also common, however. Astrophysical plasmas often consist of a mixture of dominant species, as well as containing trace elements.1,13 Disruptions,2 which are a common cause of electron runaway in tokamaks, are also typically associated with an increase in impurity content, either due to deliberate gas injection for mitigation purposes or due to plasma-wall interaction. Therefore, in Ref. 14, the initial value formulation of the problem posed in Ref. 9 was extended to account for arbitrary plasma composition. The potential for ions accelerated during a disruption to excite low frequency plasma instabilities was considered analytically. The results were inconclusive since the asymptotic expansion used to develop the analytic solution was not strictly valid for disruption-type parameters. The limitations of the analytic solutions available in previous work motivate the development of a numerical tool to allow detailed study of the time evolution of an ion runaway distribution.
Here, we describe the formulation and implementation of an efficient finite-difference–spectral-method tool, CODION (COLlisional Distribution of IONs), that solves the two-dimensional momentum space ion kinetic equation in a homogeneous plasma. CODION solves the ion Fokker-Planck equation as an initial value problem and allows for time-variation of the electric field and bulk distribution parameters (temperature, density, charge, and mass) of each plasma species independently. Due to its speed, it is highly suitable for coupling within more expensive calculations, e.g., studies of instabilities driven by the fast ions or comprehensive modeling of ion acceleration with self-consistent coupling to solvers of Maxwell’s equations. Using CODION, we obtain illustrative two-dimensional ion velocity space distributions, which demonstrate the typical behaviour of runaway ions in a variety of physical scenarios. We show that during typical tokamak disruptions, ions are unlikely to be accelerated to velocities high enough for resonant interaction with toroidal Alfvén eigenmodes (TAEs). Therefore, the experimentally observed TAE activity cannot be explained by the ion runaway mechanism alone.

The rest of the paper is organized as follows. In Sec. II, we describe the ion Fokker-Planck equation, and in Sec. III we outline its numerical implementation in CODION. In Sec. IV, we explore the numerical solution, including the effect of various approximations to the collision operator. In Sec. V, the model is applied to a variety of physical scenarios, illustrating typical acceleration time scales in laboratory and space plasmas. Finally, we close with concluding remarks in Sec. VI.

II. RUNAWAY ION DISTRIBUTION

We consider the problem of ion acceleration by induced electric fields with a component parallel to the background magnetic field in a plasma. We restrict ourselves to straight field line geometry and assume a homogeneous background plasma. The time evolution of the ion distribution is then given by the Fokker-Planck equation, and particle acceleration will be opposed by various friction forces. The effect of friction with neutral particles can be significant in various physical situations, for example, giving rise to charge exchange losses, which was studied in the context of lightning discharges in Ref. 8. Here, we will focus on fully ionized quasi-neutral plasmas, in which case the friction is the result of inter-species Coulomb collisions only.

We are interested in the initial value problem where an electric field appears in what was previously an equilibrium state. Therefore, we assume that the initial particle distribution functions are stationary Maxwellians $f_\text{eq}$—possibly at different temperatures—and consider their distortion from this state by the electric field. We linearize the collision operator about this background Maxwellian, and neglect the non-linear contribution to the evolution. This restricts the study to situations where only a small fraction of the ion population is accelerated, or to the initial stages of ion runaway. Once a high energy population forms, the runaway ions have the potential to excite instabilities, which will have a strong impact on the further evolution of the distribution. Note that the non-linear terms of the kinetic equation are sometimes required to account for the transfer of energy from the electric field into heating the distribution. These aspects of the longer term evolution of the distribution are beyond the scope of the work presented here.

The linearized collision operator for self-collisions is given by $C_{ii}(\{ f \}) = C_{ii}(\{ f \}) + C_{ii}(\{ f \})$, as described in Ref. 9, where the first term, the test-particle operator, describes collisions of the perturbed distribution with the bulk, while the second term, the field-particle operator, describes the response of the bulk to the perturbation. We will follow the approach described in Ref. 15 and approximate the field-particle operator with restoring terms such that the collision operator satisfies momentum and energy conservation, non-negative entropy production, and vanishes for a perturbed Maxwellian—properties which are known to be satisfied by the exact operator.

Collisions with the other ion species could be treated similarly, however this would require the simultaneous evolution of the distribution functions of multiple species. Therefore, we consider only the evolution of the ion species with the highest runaway rate, so that the other ion species remain approximately stationary and only the test-particle piece of the unlike ion collision operator needs to be retained. While it is difficult to verify a priori that this condition is satisfied, it can be determined by numerical simulation of each ion species individually, assuming the others to remain stationary. Due to the sensitivity of the acceleration rate to ion charge and mass (as demonstrated in Sec. V), the condition can typically be well satisfied as the acceleration rate of different species is often separated by several orders of magnitude.

The velocity-dependent friction on a test particle resulting from collisions with a Maxwellian background species has a peak near the thermal velocity of the background due to the form of the Coulomb interaction between charged particles. In the case of electrons, the friction force will be a monotonically decreasing function for velocities above the electron thermal velocity, allowing an electron to run away to large energy (where relativistic and synchrotron effects become dominant). We focus on situations where the ion, $i$, and electron, $e$, thermal velocities satisfy $v_{Te} \gg v_{Ti}$, meaning that their temperatures are sufficiently similar that $T_e/T_i \gg \sqrt{m_e/m_i}$. Then, if an ion is accelerated away from the bulk, friction against the electron population will increase with velocity. This has the consequence that the ion acceleration will be naturally balanced by the electron friction for some $v < v_{Te}$, if the electric field is below a threshold value similar to the Dreicer field for electrons.

Since an electron reacts to the electric field on a time-scale $m_e/m_i$ times shorter than the ions, we assume the electron population to be in a quasi-steady state on all time-scales of interest for ion acceleration. Parallel force balance for the electron distribution then requires that the total electron-ion friction cancels the acceleration by the electric field, $neE_y = \sum_j R_{ij} = \sum_j \int d^3v m_i v_i C_{ij}$, where the sum is taken over all ion species $j$ in the plasma. Due to the small mass ratio, the electron-ion interaction is dominated by pitch-angle scattering, so that $C_{ij} = C_{ee} n_i Z_j^2 / n_i Z_i^2$, and we
can solve for the friction between electrons and the ion species of interest, \( R_{\text{ie}} = (n_i Z_i^2 / Z_{\text{eff}}) eE_i \), where the effective charge is \( Z_{\text{eff}} = \sum n_i Z_i^2 / n_i \).

The electron distribution can be written as \( f_e = f_{\text{Me}} + f_{\text{t}} \), with \( f_{\text{Me}} \) a Maxwellian distribution drifting with the bulk ion velocity \( \mathbf{V}_i \), and a correction \( f_{\text{t}} \) varying over velocities of order \( v_{\text{pe}} \), accounting for the electron drift behavior in the electric field. Then, the linearized ion-electron collision operator, neglecting terms quadratic in \( f_{\text{t}} \), can be simplified,\(^\text{1,18}\) noting that momentum conservation in binary collisions requires that \( \mathbf{R}_{\text{ie}} = -\mathbf{R}_{\text{ei}} \).

\[
C_{\text{ie}} \{ f_{\text{t}}, f_e \} = \frac{R_{\text{ei}}}{m_i v_i} \frac{\partial f_i}{\partial v} + C_{\text{ei}} \{ f_i, f_{\text{Me}}(v - \mathbf{V}_i) \}. \tag{1}
\]

The first term, which describes the friction arising from the drifting electron distribution and was calculated above, readily combines with that describing acceleration by an electric field, giving rise to the effective electric field \( E^* = E_i - R_{\text{ei}} / n_i Z_i e = (1 - Z_i / Z_{\text{eff}}) E_i \). Thus, as noted in the Introduction, in a pure plasma where \( Z_i = Z_{\text{eff}} \), net ion acceleration will not occur. Light ions with \( Z_i < Z_{\text{eff}} \) can be accelerated in the direction of the electric field. Heavy impurities with \( Z_i > Z_{\text{eff}} \) will be accelerated in the opposite direction, which is in the direction of electron runaway (the latter case was studied by Gurevich\(^\text{11}\)). The second term in Eq. (1) describes the slowing down of the fast ions on the electrons, as well as the slow energy exchange between the bulk species. Note that the ion flow velocity correction is time dependent and formally small in the runaway density. This term will become more significant in the ion-electron momentum exchange as the runaway distribution builds up.

Finally then, the kinetic equation we consider for the evolution of the ion distribution function in the presence of an accelerating electric field and arbitrary plasma composition is the following:

\[
\frac{\partial f_i}{\partial t} + Z_i e E^* \left( \frac{\partial}{\partial v} + \frac{1 - v^2}{v^2} \frac{\partial}{\partial \xi} \right) f_i = \sum_s C_{\text{us}} \{ f_i \}, \tag{2}
\]

where \( \xi = v_{\parallel} / v \), and the effect of collisions with the background Maxwellian populations is described by the sum over all particle species \( s \) in the plasma

\[
\sum_s C_{\text{us}} \{ f_i \} = \frac{1}{\tau_{\text{us}}} \sum_s n_s Z_s^2 \frac{\phi(x_s) - G(x_s)}{2x_s^2} \left( 1 - \frac{x_s^2}{2} \right) \frac{\partial f_i}{\partial x_s^2} + \frac{1}{x_s} \frac{\partial}{\partial x_s} \left( \frac{T_i}{T_s} x_s^2 G(x_s) f_i + x_s G(x_s) \frac{\partial f_i}{\partial x_s} \right) + \frac{1}{\tau_{\text{us}}} \frac{2v_i(x_s) v_{\perp} f_{\perp}}{v_E(x_s) v_{\perp} f_{\perp}}, \tag{3}
\]

where \( \tau_{\text{us}}^{-1} = n_s e^2 Z_s^2 \ln A / 4 \pi e^2 m_i v_{\perp}^2 x_s, x_s = v_{\perp} / v_s = \sqrt{m_i v_{\perp}^2 / 2 T_s} \) and the usual error function \( \phi(x) = (2 / \sqrt{\pi}) \int_0^x dy e^{-y^2} \) and Chandrasekhar function \( G(x) = \phi(x) - x \phi'(x) / 2 \) appear. The dimensionless moments \( u_i \) and \( Q \) of the distribution function appearing in the momentum and energy restoring terms of the self-collision operator are

\[
u_i(v) = 4 \frac{G(x_i)}{x_i}, \quad \nu_E(v) = 2 \left( 4 \frac{G(x_i)}{x_i} - \phi(x_i) \right). \tag{5}
\]

The runaway behavior of interest can be demonstrated by considering the simpler dynamics of an ion test particle\(^\text{8,10}\) in the presence of the electric field. The ion equation of motion takes the form \( m_i \frac{d v_{\parallel}}{dt} = Z_i e E^* + F^\text{res} \), where the collisional friction on a test particle is given by

\[
F^\text{res}(v) = -Z_i^2 e E_D \sum_s \frac{n_s Z_s^2 v_s}{n_i v_i} \left( 1 + \frac{m_i}{m_s} \right) G(x_s), \tag{6}
\]

and \( E_D = n_s e^3 \ln A / 4 \pi e^2 T_s \) is the Dreicer field. Thus, ion acceleration can occur when \( E_i / E_D > F^\text{res} / Z_i e E_D (1 - Z_i / Z_{\text{eff}}) \). The required electric field values are illustrated in Fig. 1, for low and high effective charge. The figures illustrate the non-monotonocity of the ion friction force, with one maximum near the thermal velocities of the ion species, and another near the electron thermal velocity. Therefore, as first described by Furth and Rutherford,\(^\text{12}\) for a sufficiently strong electric field we may expect ions to be accelerated from their initial velocity to a higher velocity at which friction on the electrons dominates, giving rise to a suprathermal population in the plasma. We will compare the characteristics of the test-particle behavior, governed by the friction force illustrated in Fig. 1, to the numerical solution of Eq. (2) in Sec. III.

Figure 1 also illustrates that the electric fields needed to accelerate ions are highly sensitive to ion charge and plasma composition, due to their effect on the effective electric field \( E^* = (1 - Z_i / Z_{\text{eff}}) E \). Note that the electric fields needed to accelerate ions beyond the electron thermal velocity are significantly larger than the minimum electric field necessary for acceleration. Such strong fields will not be considered here, since the validity of the linearization typically breaks down before the ions reach a significant fraction of the electron thermal velocity.

III. CODION

In this section, we outline the implementation of Eq. (2) in the numerical tool CODION, which solves the ion Fokker-Planck equation numerically as an initial value problem to give the evolution of the ion distribution function in the presence of an accelerating electric field. It uses a continuum-spectral discretization scheme based on that used in CODE.\(^\text{20}\) Illustrative solutions for a tokamak-like plasma are presented, and a comparison of the obtained distribution function is made with the behavior predicted by test-particle equations, demonstrating the importance of collisional diffusion for the runaway of ions. In addition, we investigate the
effect of different choices for the self-collision field-particle operator.

The pitch-angle dependence of the distribution function is represented by a truncated Legendre polynomial expansion, while velocity is discretized on a uniform grid $v = v_n = n \Delta v, n = 0, 1, ..., N - 1$:

$$f_l(v_n, \zeta, t) = \sum_{l=0}^{l_{\text{max}}} f_l(v_n, t) P_l(\zeta),$$

(7)

where the Legendre polynomials $P_l$ obey the orthogonality relation $\int_{-1}^{1} d\zeta P_l(\zeta) P_m(\zeta) = \delta_{l,m}/2(2l + 1)$, and

$$f_l(v_n, t) = \frac{2l + 1}{2} \int_{-1}^{1} d\zeta P_l(\zeta) f_l(v_n, \zeta, t).$$

(8)

The integral operation $(L + 1/2) \int_{-1}^{1} d\zeta P_l(\zeta)$... is applied to the kinetic equation in Eq. (2) for each $L$, producing a linear set of equations for the quantities $f_l(t, t)$, using the boundary condition $f_{l_{\text{max}}}(v, t) = 0$ for all $t$. Well-known recurrence relations for the Legendre polynomials are used to obtain analytic expressions for all the terms appearing in the equation. For example, the Legendre polynomials are eigenfunctions of the linearized collision operator, while the electric field-term will produce a coupling between $f_l$ and $f_{l+1}$ modes. This procedure exactly captures number conservation for any choice of $l_{\text{max}} > 1$. The velocity derivatives appearing in the kinetic equation are represented with five-point stencils

$$\left. \frac{\partial f_l}{\partial v} \right|_{v_n} = \frac{1}{12 \Delta v} \sum_{m=0}^{N-1} (-\delta_{l,m-2} + 8 \delta_{l,m-1} - 8 \delta_{l,m+1} + \delta_{l,m+2}) f_l(v_m),$$

(9)

and

$$\left. \frac{\partial^2 f_l}{\partial v^2} \right|_{v_n} = \frac{1}{12 \Delta v^2} \sum_{m=0}^{N-1} (-\delta_{l,m-2} + 16 \delta_{l,m-1} - 30 \delta_{l,m} + 16 \delta_{l,m+1} - \delta_{l,m+2}) f_l(v_m),$$

(10)

formally introducing an error of order $O(\Delta v^3)$. The integral moments of the ion distribution appearing in the self-collision model operator are discretized with a quadrature of the form $\int dv A(v) = \sum_m w_m A(v_m)$, where the quadrature weights $w_m$ are chosen according to Simpson’s rule, also with error of order $O(\Delta v^3)$.

For the distribution function to be single-valued and smooth at the origin, we enforce the boundary condition $f_l(0) = 0$ for all $l > 0$ and $df_l/df_{l-1}(v=0) = 0$. Since we restrict ourselves to cases where electron friction will dominate the electric field at sufficiently high velocities, the maximum resolved velocity can always be chosen so that only insignificant numbers of particles are near the edge of the grid, minimizing the effect of the choice of boundary condition. We use the Dirichlet boundary condition $f_l(v_N) = 0$ for all $l$ at the maximum velocity. A detailed investigation of the convergence properties of solutions with respect to discretization parameters is described in Ref. 21.

The discretization procedure outlined above casts the kinetic equation, Eq. (2), into the form

$$\frac{\partial F}{\partial t} + MF = 0,$$

(11)

where $M$ is a sparse matrix and $F$ is a vector representing the discretized distribution function. Time integration is performed with the first order implicit scheme

$$F(t_{n+1}) = [I + \Delta M(t_{n+1})]^{-1} F(t_n),$$

(12)

where any time-dependence of the operator $M$ is due to time-variation of electric fields and background plasma parameters. An arbitrary plasma composition is determined by a set of input vectors containing particle masses $m_z$, corresponding charge states $Z_z$, charge densities $\rho_z$, and temperatures $T_z$.

Figure 2(a) shows a typical example of the evolution of the ion distribution for a case where the electric field is above the minimum required for runaway acceleration. The plasma parameters used are characteristic for tokamaks with
a hot bulk deuterium plasma at 1 keV and fully ionized native carbon impurities. Note that the loop voltage is typically \( \leq 1 \) V in normal tokamak operation, corresponding to \( E \approx 0.2 \) V/m.\(^2\) A contour plot of the distribution in velocity space when steady state is reached is shown in Fig. 2(b). For the parameters used here, approximately 5% of the ion population has been accelerated and the linearization used to derive Eq. (2) is well satisfied.

The steady state distribution is typically established in 10–20 s at this temperature and density, and the time-scale varies with plasma parameters as the collision time defined in connection with Eq. (3), \( \tau_\text{c} \propto T_i^{3/2}/n_c \). For stronger electric fields, the initial evolution of the distribution can be followed, but the linearization breaks down before the steady state can be reached. Numerical simulations indicate that the entire ion distribution will eventually run away when \( E \approx 0.2E_D \) for the \( Z_{\text{eff}} = 2 \) case considered here (this will vary with species and composition as indicated in Fig. 1), and the linearization breaks down within \( \sim 30 \) ms with such an electric field.

As discussed in the Introduction, approximate analytic solutions of the ion kinetic equation can be found in the literature. However, their potential for use in benchmarking is severely limited. The initial-value problem was considered in Refs. 9 and 14, but as noted in the derivations the initial-value problem was considered analytically in Ref. 12 provides the magnitude of the distribution in the runaway region, as a function of electric field; however, it is given in closed form only in the case of a strong field. A direct comparison is therefore not possible in the case of Fig. 2, where the long-time evolution of the distribution due to realistic electric fields, near the critical value needed to produce runaway, was considered. A further discussion of the comparison to the analytic solutions may be found in Ref. 21, but it is not pursued further here.

**IV. RESULTS**

Expansions of the collision operators appearing in Eq. (2) have previously been used to consider ion runaway analytically. Here, we compare the effect of various approximations to the collision operator on the numerical solution of the distribution function. The plasma parameters of Fig. 2 are used as a basis in the comparisons, but the behavior illustrated is characteristic of a wide range of parameters. We first consider the effect of neglecting the conservative field-particle terms in the self-collision operator. Figure 3 shows the \( \xi = 1 \) cut of the distribution of the bulk deuterium population for two cases, with effective charge \( Z_{\text{eff}} = 1.5 \) and \( Z_{\text{eff}} = 2 \), respectively. It can be seen that, using only the test-particle operator, the dominant behavior of the fast-ion population as given by the fully conserving case is reproduced. This is expected, since the conserving field-particle terms are proportional to \( f_{\text{m0}} \propto \exp(-x^2) \), and therefore act mainly on the thermal bulk of the distribution. The main difference is that the conservative operators typically lead to a runaway rate which is at least twice as large, due to the parallel momentum they inject back into the low-energy part of the distribution.

Figure 3(a) shows a case with \( Z_{\text{eff}} = 1.5 \). With a lower amount of impurities to which momentum and energy can be transferred, the fully conserving linearized collision operator for self-collisions will exhibit unphysical behavior before a significant runaway population has time to form, which is clearly illustrated by the distortion near \( v = 0 \). The reason is that when the conserving terms are kept in the kinetic equation, the distribution is heated by the electric field, causing the linearized equation to break down after some time. This is also observed in the solution obtained using only the energy conserving term, albeit less pronounced. The distribution functions obtained with the momentum conserving self-collision operator tend to stay regular for longer. Figure 3(b)
shows a similar case but with higher impurity content, $Z_{\text{eff}} = 2$, for which all operators yield well-behaving results. An additional consequence of the low effective charge was demonstrated in Fig. 1(a), where impurity ions were shown to be more easily accelerated by an electric field than the bulk species, implying that for low $Z_{\text{eff}}$ the assumption of stationary impurity ions may be violated.

In conclusion, the high-energy part of the ion distribution obtained using only the test-particle operator is in qualitative agreement with the result obtained with conservative operators, but the runaway rate is expected to be lower in the test-particle case. A quantitative investigation of runaway rates for impurity species is presented in Sec. V.

It is instructive to compare the behavior of the numerical solution to the characteristic behavior indicated by the test particle friction given in Eq. (6). Noting that the runaway ion velocity satisfies $v_{\text{r}} \ll v \ll v_{\text{cr}}$, we can expand the contributions to Eq. (6) using the known low and high-velocity limits of the Chandrasekhar function. The test particle friction in this limit reduces to

$$F_{\text{test}} = m_i v_{\text{r}} \frac{T_{\text{e}}}{T_{\text{i}}} \left[ \frac{Z_{\text{eff}} + \bar{n}}{x_i} + \frac{4}{3\sqrt{n}} \left( \frac{T_{\text{i}}}{T_{\text{e}}} \right)^{3/2} \frac{m_e}{m_i} \right],$$  \hspace{1cm} (13)

where $\bar{n} = \sum_j n_i Z_i^2 m_j/n_i m_i$ allows for arbitrary impurity content. Consider first the minimum value of the magnitude of the collisional friction force; this will determine the minimum electric field which can accelerate a fast test ion. Minimizing Eq. (13) yields

$$v_{\text{min}} = v_{\text{r}} \left[ \frac{3}{2} \frac{m_e}{m_i} \left( \frac{T_{\text{e}}}{T_{\text{i}}} \right) \left( Z_{\text{eff}} + \bar{n} \right) \right]^{1/3} ,$$  \hspace{1cm} (14)

$$F_{\text{test}}(v_{\text{min}}) = -2 m_i v_{\text{r}} \frac{T_{\text{e}}}{T_{\text{i}}} \left[ \frac{3}{2} \frac{m_e}{m_i} \left( Z_{\text{eff}} + \bar{n} \right) \right]^{1/3} .$$  \hspace{1cm} (15)

From this it follows that the minimum, “critical,” value $E_{\text{crit}}$ of the electric field above which a test ion can be accelerated is given by

$$E_{\text{crit}} = \frac{Z_i}{Z_{\text{eff}} + \bar{n}} \left( \frac{Z_{\text{eff}} + \bar{n}}{1 - \frac{Z_i}{Z_{\text{eff}}}} \right) \left( \frac{3}{2} \frac{m_e}{m_i} \right)^{1/3} .$$  \hspace{1cm} (16)

By taking $Z_i E^* + F_{\text{test}} = 0$, we can find the range of test ion velocities, $v_{\text{r1}} < v < v_{\text{r2}}$, for which acceleration in a given electric field occurs, as discussed in Refs. 1 and 12. Using the expression for the friction given in Eq. (13) results in a third order equation, however simpler approximate formulae can be obtained by noting that $v_{\text{r1}}$ will fall near to the region dominated by ion friction, and $v_{\text{r2}}$ in the region dominated by electron friction. Retaining only the corresponding terms in Eq. (13) yields, for arbitrary impurity content,

$$v_{\text{r1}} = v_{\text{r}} \left[ \frac{Z_i}{2} \frac{T_{\text{e}}}{T_{\text{i}}} \left( E \right)^{-1} \frac{Z_{\text{eff}} + \bar{n}}{1 - \frac{Z_i}{Z_{\text{eff}}}} \right] .$$  \hspace{1cm} (17)

$$v_{\text{r2}} = v_{\text{r}} \left[ \frac{3}{2} \frac{m_e}{m_i} E \right] \left( \frac{Z_{\text{eff}} + \bar{n}}{1 - \frac{Z_i}{Z_{\text{eff}}}} \right) .$$  \hspace{1cm} (18)

These equations generalize the corresponding expressions in Ref. 1 to arbitrary plasma composition. Note that these formulae are only valid when $E$ is sufficiently large compared to $E_{\text{crit}}$, since at $E = E_{\text{crit}}$ ion and electron friction contribute equally. We may expect that in steady-state, the position of the high-velocity maximum of the distribution function, denoted $v_{\text{m}}$, is close to $v_{\text{r2}}$, which scales linearly with $E$ in the approximate form given by Eq. (18). This is confirmed numerically and illustrated in Fig. 4, where we show the variation with electric field of $v_{\text{m}}$, obtained from steady-state CODION solutions of Eq. (2). Also shown are the boundary velocities of the acceleration region, resulting from numerical solution of the force balance using the full test particle friction, Eq. (6), as well as their approximate forms Eqs. (17) and (18). The values converge when the system is strongly driven by a large $E$. The linear dependence of $v_{\text{m}}$ is clearly visible at large $E$, where it approaches the value given by the test particle approximation. The analytic approximation for $E_{\text{crit}}$, Eq. (16), is only accurate to $\sim 20\%$, however, indicating...
that collisional diffusion contributes significantly to the evolution at lower electric fields. Since the linearization breaks down more rapidly at larger electric fields, it is mainly at fields near the threshold for runaway generation that the model can consistently be applied to study the long-term evolution of the ion runaway tail, making a full kinetic simulation essential for capturing the important physics. For the more massive impurities, the features of the test-particle model become increasingly accurate since the runaway ion population is further separated in velocity space from the thermal bulk.

It is important to point out that neither the diffusion terms nor the field-particle self-collisions have been accounted for in the derivations of the above estimates, which are meant to give simple expressions that show how the essential quantities scale with the plasma parameters, and to provide a useful physical picture for illustrating the ion runaway phenomenon. A complete description will be provided only by numerical solution of the kinetic equation.

V. APPLICATIONS

In this section, CODION is applied to calculate runaway ion distributions for typical solar flare and fusion plasmas. The time it takes for a fast ion population to form due to the runaway mechanism is determined, and it is investigated whether the difference in acceleration rate between different ion species can explain the enhanced abundance of heavy ions in the solar wind. We also consider the possibility of Alfvénic instabilities being driven by runaway ions during tokamak disruptions.

Throughout this section, time-scales are chosen so that significant fast ion populations have time to form, which typically takes between a few hundred to a few thousand ion-electron collision times. We define the runaway density, 

\[ n_r = \frac{1}{n_i} \int_{v > v_{0,\text{run}}} d^4v f_i, \]

as the fraction of ions with velocity larger than the low-velocity numerical solution of \( \frac{ZeE^* + F_i^{\text{test}}}{E_{\text{ci}}} = 0 \), denoted \( v_{r,1} \), which if \( E < E_{\text{ci}} \) is taken as the velocity \( v_{\text{max}} \) minimizing the friction \( F_i^{\text{test}} \).

A. Ion acceleration in solar flares

Solar flares are thought to be initiated by reconnection in the corona,

but the origin of the observed fast ion populations in flares is still not completely understood.

Both stochastic acceleration by waves and the direct acceleration of the particles by the electric field have been considered, and it appears likely that a combination of the two can be at work.

The effective accelerating field experienced by a given species varies with its charge and the effective charge of the plasma, as discussed in Sec. II. This can give rise to preferential acceleration of heavier elements under certain circumstances, and this effect was considered in Ref. 1, where estimates of the runaway rate were given based on the approximate formula of Ref. 11. With CODION we can determine the time evolution of the ion distribution function numerically, and evaluate the dependence of the runaway population on various ion parameters.

The composition of the solar plasma, particularly the metallic elements, has been studied extensively in recent years, however much uncertainty remains. We will choose parameters consistent with the choices made by Holman.

We use the plasma temperature \( T = 700 \text{ eV} \) for all particle species, and hydrogen density \( n_{\text{H}} = 3 \times 10^{17} \text{ m}^{-3} \). Elements with atomic number \( Z \leq 6 \) can readily be assumed to be fully ionized at this temperature. The plasma composition is based on the ion abundance recommended by Schmelz et al. We use a helium population of density \( n_{\text{He}}/n_{\text{H}} = 6\% \), and represent all heavier impurities by a carbon population of density \( n_C/n_{\text{H}} = 0.1\% \), corresponding to an effective charge \( Z_{\text{eff}} = 1.13 \). Electric field strengths in solar flares are not well constrained by experimental observation, and we will investigate the rate of acceleration at a range of values. The Dreicer field is \( E_D = 224 \text{ mV/m} \) for this set of plasma parameters.

The critical electric fields \( E_{\text{ci}} \) for the ion species in such a plasma are \( E_{\text{ci}}(\text{He}) = 154 \text{ mV/m} \) for hydrogen, \( E_{\text{ci}}(\text{He}) = 40 \text{ mV/m} \) for helium, and \( E_{\text{ci}}(\text{C}) = 20 \text{ mV/m} \) for carbon. Note that the acceleration rate depends strongly not only on \( E/E_{\text{ci}} \) but also on \( v_{r}/v_T \).

Figure 5 shows the \( v_{r} = 0 \) cut of the distribution functions of hydrogen (\(^1\text{H}\)), helium (\(^4\text{He}\)), and carbon (\(^{12}\text{C}\)) after 30 s of acceleration from initial Maxwellians, with the plasma parameters specified above and an electric field \( E = 50 \text{ mV/m} \). At this electric field, the critical velocities are given by \( v_{r} = v_{r,1}/v_T = 0.15 \) for hydrogen (coinciding since \( E < E_{\text{ci}}(\text{H}) \), \( v_{r,1}/v_T = 0.08 \) and \( v_{r,2}/v_T = 0.21 \) for helium, and \( v_{r,1}/v_T = 0.04 \) and \( v_{r,2}/v_T = 0.48 \) for carbon. The electric field is significantly below the hydrogen critical field, and no hydrogen runaway runaway populations form. Runaway ion populations of both helium and carbon do form however, with runaway densities \( n_{\text{He}} = 0.037\% \) and \( n_{\text{C}} = 18\% \), respectively.

FIG. 4. Electric field dependence of the location \( v_{\text{m}} \) of the maximum in the runaway tail, obtained using CODION, for a fully ionized impure deuterium plasma with \( n_C/n_D = 4\% \), \( Z_{\text{eff}} = 2.5 \) and equal temperature for all particle species. The boundary velocities \( v_{r,1} \) and \( v_{r,2} \) of the acceleration region obtained numerically from Eq. (6) (using the velocity which minimizes the friction when \( E < E_{\text{ci}} \)), and their approximate values \( v_{r,1} \) and \( v_{r,2} \) given by Eqs. (17) and (18), are also shown. The quantities shown are independent of electron density and temperature. For electric fields \( E < 0.8 E_{\text{ci}} \), no maximum forms in the runaway ion distribution.

\[ \frac{V}{V_{\text{TD}}} \]

\[ E/E_{\text{CD}} \]

\[ v_{r} \]

\[ v_{r,1} \]

\[ v_{r,2} \]

\[ v_{r,1} \]

\[ v_{r,2} \]

\[ v_{r,1} \]

\[ v_{r,2} \]

\[ v_{r,1} \]

\[ v_{r,2} \]
Positive values of $v_i$ represent the direction of the electric field. Therefore, Fig. 5 also illustrates how heavier ions correspond 2D carbon distribution function is shown in Fig. 6, displaying a strong directional anisotropy (compare to Fig. 2(b)). This can be understood by the observation that the accumulation velocity $v_{\|}$ is located at a higher value of $x_i = v_t/v_{\|}$ for heavier ions. Since pitch angle scattering of the energetic heavy ions scales with velocity like $Z_{\text{eff}}/x_i^3$, the mechanism will be less effective than for light bulk ion species in increasing the perpendicular energy of the distribution.

We will now investigate how the runaway population varies with electric field strength for different ion species. To determine the runaway density of heavier ions, we have introduced trace amounts of each ion species with charge between 2 (helium) and 18 (argon), assumed to be fully ionized. In practice, ions of charge $Z_i > 8$ will typically not be fully ionized at the temperature considered due to their high ionization energy, meaning that the results shown here will overestimate the acceleration rate of the heavier ions. The ion masses have been set to that of the most common isotope, i.e., $^7\text{Li}$, $^9\text{Be}$, $^{20}\text{Ne}$, etc. Both $^3\text{He}$ and $^4\text{He}$ are shown, with $^3\text{He}$ showing significantly enhanced runaway compared to $^4\text{He}$, for all values of $E$.

Figure 7(a) shows how the runaway density $n_r$ depends on electric field after 1 s of acceleration for various ion species present in the solar flare plasma. The figure illustrates how the average runaway rate is sensitive to ion parameters; at low electric fields, the heavier ions tend to be accelerated more slowly than light ions, while at higher fields they are the most readily accelerated. Note that above the critical electric field, $E_{ci} \approx 40 \text{ mV/m}$ for helium, the runaway population increases significantly faster than for $E < E_{ci}$. The average runaway rate of $^3\text{He}$ is seen to be orders of magnitude higher than that of $^4\text{He}$ for all electric fields considered.

Finally, we illustrate the dependence of the average acceleration rate on ion charge and mass. Figure 7(b) shows the runaway density $n_r$ after 1 s of acceleration as a function of ion charge $Z_i$ for various electric fields. Ions of charge between $Z_i = 4$ ($^8\text{Be}$) and $Z_i = 8$ ($^{16}\text{O}$) are seen to be preferentially accelerated over lighter or heavier elements for low electric fields. For $Z_i > 8$, the trend depends on electric field. For low electric fields, the runaway population decreases...
with charge, while for larger electric fields (in this case above approximately 50 mV/m), heavier ion species may be accelerated more readily than the lighter species. Further studies are required to determine the full effects of variation in background composition, temperature, density, and charge states on the relative rates of acceleration between different ion species. The results presented here demonstrate the utility of CODION for the problem.

As previously noted, acceleration by quasi-static electric fields is not the only mechanism for ion acceleration in a flare plasma. Interaction with Alfvén waves can accelerate ions which have velocities above the Alfvén velocity. As this is usually well above the thermal ion velocity, an initial acceleration by electric fields may be required before the process becomes significant. CODION provides the means for more accurate modeling of the effects of such interactions.

B. Tokamak disruptions

During tokamak disruptions the plasma temperature drops from the typical operating regime of around several keV to a few eV in a couple of milliseconds. A large electric field is initially induced parallel to the magnetic field to maintain the plasma current of several MA, potentially leading to the formation of a beam of energetic electrons through the runaway mechanism. The potential for damage by such a focused high-energy beam on contact with the vessel wall is large, and runaway generation must as far as possible be suppressed. To study the physics and mitigation of runaway electrons, disruptions can be induced by the injection of large quantities of noble gas, often in amounts comparable to the initial plasma inventory or larger.2

The large induced electric field will usually decay rapidly on a timescale of a few ms in response to the formation of a narrow runaway electron (RE) beam. With runaway electrons reaching energies of order tens of MeV, they can carry a significant fraction of the pre-disruption plasma current and can drive high frequency electromagnetic instabilities through resonant interactions.27–30 Recently, low-frequency magnetic fluctuations in the range of \( f \approx 60 \) – 260 kHz have been observed in the TEXTOR tokamak during induced-disruption studies with argon massive gas injection (MGI). These fluctuations take the form of either a strong signal at a distinct frequency31 or accompanied by broadband activity.32 The fluctuations appear to limit the RE beam formation in these cases, as the magnetic perturbations may scatter the runaway electrons and provide passive mitigation. Aside from the potential consequences for mitigation, observed instabilities offer a non-intrusive diagnostic for the inverted region of their energy distribution, \( \frac{df}{dE} > 0 \) where \( E = \frac{m_0 c^2}{2} \) is the particle energy, can reach the lower Alfvén resonance, \( v_0 = v_A \) and may drive the TAE. If the radial runaway ion profile peaks on axis, the spatial gradient \( \partial f / \partial r \) will give an additional positive contribution to the growth rate. Taking parameters characteristic of argon MGI-induced disruptions, \( n_d = 3 \times 10^{19} \) m\(^{-3} \), \( n_{Ar} = 0.1 n_D \), \( Z_{Ar} = 2 \), and anticipating a native background carbon impurity with \( n_{C} = 0.08 n_D \) and \( Z_C = 2 \), so that \( Z_{eff} = 1.26 \), a background ion temperature of \( T_i = 10 \) eV, toroidal magnetic field \( B = 2 \) T and major radius \( R_0 = 1.75 \) m means that the resonance condition \( v_{||} = v_A / 3 \) requires deuterium ions with velocities \( v \approx 35 T_D \).

The electric field required to accelerate bulk ions to the resonant velocity at these low temperatures is substantial, varying in response to changes in \( Z_{eff} \) but is typically \( \geq 0.3 E_D \) \( \approx 100 \) V/m. Such field strengths are unlikely to occur during a disruption, and they would be short-lived if they did.37 Therefore, we conclude that whilst ion runaway may be of interest in hot fusion plasmas, runaway ions are unlikely to provide the drive for the observed fluctuations during disruptions.

To quantify the electric field needed for significant ion runaway, we show in Fig. 8 how the deuterium distribution evaluated after 2 ms of acceleration from an initial Maxwellian—a typical time scale for the induced electric field—varies with (a constant) electric field. The parameters are \( n_d = 3 \times 10^{19} \) m\(^{-3} \), \( T_i = 10 \) eV, and the same plasma composition as before with \( Z_{eff} = 1.26 \). It is seen that for electric fields below \( \approx 200 \) V/m, no runaway tail tends to form, and even with \( E = 260 \) V/m the fast ions are far from the resonant velocity near \( 35 T_D \). The behavior is sensitive to which temperature is chosen for the plasma: increased temperature decreases the electric fields needed to accelerate ions, but makes the acceleration timescale longer.

Note that a higher amount of assimilated argon, or a weaker magnetic field, would lead to a lower Alfvén velocity and TAE frequency, and therefore the runaway ions would reach the resonance condition more easily. The higher
electron density leads to an increased collision frequency, however, so that the runaway ion distribution requires a longer time to form. The level of argon absorption during mitigation varies between machines\(^2\) and the absorption appears to decrease for bigger machines. Therefore, although we may expect a higher electric field in large tokamaks such as ITER, the higher bulk plasma density and likely lower absorption suggests no ion runaway will occur. Note that as the ITER magnetic field is so high, even if the assimilated argon was equal to the initial deuterium inventory, the ions would need to reach \(\sim 40\tau_{T1}\) (at 10 eV) to reach the typical \(v_A/3\) resonance, assuming \(B = 5\) T, \(R_0 = 6\) m, and \(n_D = 1 \times 10^{20}\) m\(^{-3}\).

Finally, we note that the model presented in this paper assumes the initial ion distribution to be a stationary Maxwellian. Fast ion populations present due to heating schemes in use before the disruption may not be completely expelled, and it is not certain that the initial distribution is accurately described by a Maxwellian. The dynamics of ion acceleration starting from these non-Maxwellian distributions may yield a fast-ion population more readily. Furthermore, since plasma conditions change on a time scale of a few collision times during the sudden cooling associated with instabilities or disruptions, so-called “hot-tail” runaway generation may occur. This has been shown to be an important effect for runaway electrons, where a seed of runaway electrons is provided by fast electrons present before cooling.\(^{38}\) These fast electrons are cooled at a slower rate than the low-energy electrons, and may find themselves in the runaway region when the plasma has reached its final temperature. However, using CODION to investigate the effect of hot-tail ions, it has been concluded that the effect is small for realistic fields. The reason is that, if the electric field is not high enough for the ions to overcome the friction and become runaways in the first place, all hot-tail ions will necessarily also be slowed down. A low-velocity inverted ion population may however form as a result of the cooling process even for such low electric fields, with its peak near the velocity which minimizes the collisional friction at the final temperature (typically around 6–8 thermal velocities). This is still significantly lower than that needed for resonant interaction with Alfvén waves.

One refinement to the model would be the inclusion of “knock-ons,” i.e., large-angle collisions, which have been neglected in the Fokker-Planck equation. It is well known that single collisions can change the momenta of the interacting particles significantly, and a runaway ion interacting with a bulk ion could cause both to end up in the runaway region. In a situation where the electric field is low enough that runaway ions are produced at a low rate through the standard acceleration mechanism, knock-on collisions could possibly contribute significantly to the runaway generation rate. This has been demonstrated to be the case for electron runaway, where this effect drastically affects the rate at which runaways are produced. A simplified runaway ion knock-on operator could potentially be constructed from the Boltzmann collision integral under the assumption that fast ions accumulate near \(v_2\) and collide mainly with the bulk distribution, since the fast ion distribution is assumed to be a small perturbation in our linearized model. However, there are differences between ion runaway and electron runaway that suggest that knock-on runaway generation is a less significant effect for ions than for electrons. Since our linearized model restricts the study to cases where \(E \sim E_{ci}\) (which is also the regime where knock-on generation would be expected to be significant), the accumulation velocity near \(v_2\) will not be significantly larger than the runaway velocity \(v_{r1}\). Therefore, collision events where both particles end up in the runaway region will be less frequent. This is in contrast to electron runaway, where the electrons have unbounded energy (neglecting radiation effects).

There are applications for knock-on operators other than avalanche generation. It has been suggested that fast ion populations due to other sources—for example, hot alpha particles created in fusion reactions or ions heated by external sources such as neutral beam injection (NBI) or radio frequency (RF)-heating—could accelerate bulk impurity ions, which could in turn be used for diagnostics.\(^{39,40}\) The suggested collision operator could be implemented in CODION, and the time-evolution of bulk impurities solved for in the vicinity of an assumed or numerically obtained background of fast ions, however this is outside the scope of the present paper.

**VI. CONCLUSIONS**

Electron runaway resulting from the occurrence of a strong electric field in a plasma has been the subject of extensive study, and numerical tools exist to simulate the electron dynamics. The analytic description of the associated ion acceleration was developed at the same time, but its application has been much more limited and is restricted by the various approximations to the collision operator which were required.
We have developed an efficient open-source numerical tool, CODION, which solves the ion Fokker-Planck equation as an initial value problem in a fully ionized plasma. A uniform background magnetic field is assumed, along with initially stationary Maxwellian distributions, however arbitrary impurity densities and temperatures may be specified. A model operator for ion self-collisions based on that used in the gyrokinetic code GS2 (Ref. 42) has been employed, satisfying momentum and energy conservation, non-negative entropy production, and self-adjointness. A simplified analytical model based on the large mass ratio is used for ion-electron collisions, allowing a description of ion-electron friction caused by the perturbation of the electron distribution due to the electric field. However, we wish to note that our model will break down for strong electric fields, as the electrons—which are assumed to be in force balance with the electric field and ion friction—will be rapidly accelerated by electric fields approaching the Dreicer field. A full description of such scenarios would require the simultaneous evolution of the ion and electron distributions, for example, by coupling the CODION and CODE codes.

The effect of various approximations to the collision operator commonly used in the literature has been studied numerically. It has been demonstrated that the addition of momentum and energy conservation in self-collisions mainly acts to increase the rate at which the fast ion population builds up, while the qualitative behavior is largely unaffected. For strong electric fields, the test particle description is seen to reproduce well the characteristic velocity achieved by the runaway population, and the predicted critical electric field for ion acceleration is accurate to ~20%. Using the test particle approximation, we derived concise analytic expressions for the critical electric field for ion runaway, Eq. (16), and the typical runaway energy, Eqs. (17) and (18), and tested these expressions against direct numerical simulation with CODION.

The output of CODION is the evolution of the 2D velocity space ion distribution. The utility of this has been demonstrated for calculating acceleration rates of ions in solar flare plasmas. The average rate at which ions are accelerated has been evaluated for a range of ion masses and charges for a solar flare scenario based on that considered by Holman, and an exponential dependence of the buildup of a runaway population with charge for \( Z > 8 \) has been illustrated for this scenario.

Low-frequency instabilities, in the range characteristic of TAE modes, have been observed in post-disruption tokamak plasmas, where the disruption was induced by massive gas injection. Using CODION, we have considered the potential for ions accelerated in the disruption-induced electric field to drive such modes resonantly. The post-disruption discharge parameters are not well constrained experimentally and simulations of a range of values indicate that ion acceleration is possible. The typical maximum ion velocity achieved is too low for resonant interaction to occur, however, and the rate of runaway generation is too slow for a significant runaway density to be reached in the short-lived electric fields of a typical disruption.

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41CODION is available at https://github.com/Embreus/CODION.
Paper E

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*The Gaussian Radial Basis Function method for plasma kinetic theory*,
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The Gaussian radial basis function method for plasma kinetic theory

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A B S T R A C T

Description of a magnetized plasma involves the Vlasov equation supplemented with the non-linear Fokker–Planck collision operator. For non-Maxwellian distributions, the collision operator, however, is difficult to compute. In this Letter, we introduce Gaussian Radial Basis Functions (RBFs) to discretize the velocity space of the entire kinetic system, and give the corresponding analytical expressions for the Vlasov and collision operator. Outlining the general theory, we also highlight the connection to plasma fluid theories, and give 2D and 3D numerical solutions of the non-linear Fokker–Planck equation. Applications are anticipated in both astrophysical and laboratory plasmas.

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1. Motivation

A fundamental macroscopic description of a magnetized plasma is the Vlasov equation supplemented by the non-linear inverse-square force Fokker–Planck collision operator \[ \frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \nabla f_a + \frac{e_a}{m_a} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla \phi_a = \sum_b C_{ab} (f_a, f_b). \]

where \( f_a \) is the distribution of species \( a \) with charge \( e_a \) and mass \( m_a \). The electric and magnetic fields depend on the distribution \( f_a \) through Maxwell’s equations. This model assumes a statistical description of Coulomb interaction in the limit of small-angle scattering, with the changes in \( f_a \) due to collisions with species \( b \) described by

\[ C_{ab} = \frac{\partial}{\partial \mathbf{v}} \left[ A_{ab} f_a + \frac{\partial}{\partial \mathbf{v}} \cdot (D_{ab} f_a) \right]. \]

The friction and diffusion coefficients are given by the expressions

\[ A_{ab} = L_{ab} \left( 1 + \frac{m_a}{m_b} \right) \frac{\partial \phi_b}{\partial \mathbf{v}}, \quad D_{ab} = -L_{ab} \frac{\partial^2 \psi_b}{\partial \mathbf{v} \partial \mathbf{v}}, \]

where \( L_{ab} = (e_a e_b / m_b) \ln \Lambda_{ab} \) and \( \ln \Lambda_{ab} \) is the Coulomb logarithm which represents a physical cut-off for small-angle collisions.

The Rosenbluth potentials appearing in the friction and diffusion coefficients are weighted integrals of the distribution function

\[ \psi_b(\mathbf{x}, \mathbf{v}, t) = -\frac{1}{4\pi} \int d^3 \mathbf{v}' f_b(\mathbf{x}', \mathbf{v}', t) \frac{1}{|\mathbf{v} - \mathbf{v}'|}, \]

\[ \psi_b(\mathbf{x}, \mathbf{v}, t) = -\frac{1}{8\pi} \int d^3 \mathbf{v}' f_b(\mathbf{x}', \mathbf{v}', t) |\mathbf{v} - \mathbf{v}'|, \]

and they satisfy the velocity-space Poisson equations \( \nabla^2 \psi_b = \psi_b \) and \( \nabla^2 \psi_b = f_b \). Expressed in terms of the potential functions, the Fokker–Planck collision operator is

\[ C_{ab} = L_{ab} \left[ \frac{m_a}{m_b} f_a f_b + \mu_{ab} \frac{\partial \psi_b}{\partial \mathbf{v}} \cdot \frac{\partial f_a}{\partial \mathbf{v}} - \frac{\partial^2 \psi_b}{\partial \mathbf{v} \partial \mathbf{v}} \cdot \frac{\partial f_a}{\partial \mathbf{v}} \right]. \]

where \( \mu_{ab} = m_a / m_b - 1 \). A common approach for numerical evaluation of \( C_{ab} \) follows a two-phase method where one first inverts the velocity-space Laplacian operators and then directly evaluates the collision operator. Boundary conditions for the Poisson equations can be obtained by limiting the solution to a sub-space and evaluating the expressions at the boundary using a multipole expansion of the potentials [2], or by imposing the free-space solution outside the sub-space [3]. Another sophisticated approach is based on fast spectral decomposition as described in Ref. [4].

In this Letter, we propose a new approach using a mesh-free shifted-Maxwellian representation which is intuitively appealing and straightforward to implement. The solution thus obtained is \( C^\infty \) smooth, extends to \( \mathbf{v} \to \infty \), and allows compact representation of any interesting macroscopic quantity (number, momentum, energy density, and so on). The Letter is organized as follows: In Section 2 we present the idea for using Gaussian radial basis functions...
to solve the kinetic equation. The numerical implementation is described in Section 3 and simulations in 2D and 3D velocity space are presented in Section 4. Finally, we discuss and summarize our results in Sections 5 and 6.

2. The Gaussian RBF method

To solve the kinetic equation, Eq. (1), we write the total distribution as a finite sum of shifted Maxwellians

\[ f_a(x, v, t) = \sum_i w_{ia}(t) F_{ia}(x, v), \]

where each Maxwellian, \( F_{ia} = (\gamma_{ia}/\pi)^{3/2} \exp[-\gamma_{ia}(v - \mathbf{v}_a)^T(v - \mathbf{v}_a)], \) is normalized to unity and the weights \( w_{ia} \) allowed to evolve in time. The width parameters \( \gamma_{ia} \) and mean velocities \( \mathbf{v}_a \) can be arbitrary functions of position. The shifted Maxwellians are nothing other than Gaussian Radial Basis functions (RBFs) which have found numerous applications in applied mathematics – in particular for the construction of neural networks [5]. For compactness, in what follows we will retain the spatial dependence of all quantities but will not write the dependence explicitly.

The potential functions then take the form

\[ \varphi_a(x, v, t) = \sum_i w_{ia}(t) \varphi_{ia}(v), \quad \psi_a(x, v, t) = \sum_i w_{ia}(t) \psi_{ia}(v), \]

where the Gaussian RBF potentials \( \varphi_{ia} = -\sqrt{\gamma_{ia}} \Phi(\sqrt{\gamma_{ia}}(v - \mathbf{v}_a)) \)

\(/(4\pi) \) and \( \psi_{ia} = -(1/\sqrt{\gamma_{ia}}) \Psi(\sqrt{\gamma_{ia}}(v - \mathbf{v}_a))/(8\pi) \) are defined in terms of the functions \( \Phi(s) = \text{erf}(s)/s \) and \( \Psi(s) = [s + 1/(2s)]\text{erf}(s) + \exp(-s^2)/\sqrt{\pi} \), where \( \text{erf}(s) \) is the error function. We thus find a simple bilinear expression for the complete non-linear operator

\[ C_{ab} = \sum_{k, \ell} w_{ik}(t) w_{j\ell}(t) C_{i\ell}^{k\ell}(v), \]

where the Gaussian RBF collision tensor is

\[ C_{i\ell}^{k\ell} = \frac{1}{L_{ab}} \left[ \frac{m_a}{m_b} f_{ib} f_{ib} + \beta a b \frac{\partial^2 F_{ib}}{\partial v^2} \frac{\partial \Phi}{\partial v} - \frac{\partial^2 F_{ib}}{\partial v^2} \frac{\partial \Phi}{\partial v} \right], \]

such that \( C_{ab}(v) = 0 \). As we have analytical expressions for \( F_{ia}(v), \varphi_{ia}(v), \) and \( \psi_{ia}(v), \) the tensor \( C_{ab}(v) \) is easy to implement and fast to evaluate at any point in velocity space.

In problems with azimuthal symmetry, a 2D RBF scheme can be developed with axisymmetric ring-like RBF-basis:

\[ F_{i}(v) = (\gamma_{ia}/\pi)^{3/2} I_0(2\gamma_{ia} |v_a| \sqrt{\mathbf{v}}_a) e^{-\gamma_{ia}(v_a^2 + (v_x^2 + (v_y^2 + (v_z^2))}, \]

where \( I_0 \) is the order-zero modified Bessel function of the first kind and \( (v_x, v_y, v_z) \) are the cylindrical velocity-space coordinates. Although explicit expressions for axisymmetric RBF potentials \( \varphi_i \) and \( \psi_i \) are not available in a closed form, they can be evaluated numerically to machine precision at any requested point.

3. Collocation options

We describe two different methods for obtaining an ordinary differential equation for the time evolution of weights: the Galerkin and the center-collocation projections. In the Galerkin method, the kinetic equation – already expanded in the RBF basis – is multiplied by each individual basis function and then integrated over the entire domain. Conversely, in the center-collocation method, the kinetic equation is evaluated at the center of each RBF. Both methods yield \( N \) equations for the \( N \) RBF weights, \( w_{ia} \), and result in a differential equation for the weights that can be written in a matrix form.

For the moment, let us illustrate the method by considering the spatially homogeneous case with no electromagnetic fields. Then, the matrix equation is

\[ \sum_j M_{ij} \frac{\partial w_{ja}}{\partial t} = \sum_{b, k, \ell} w_{ja}(t) \frac{\partial w_{j\ell}(t)}{\partial v} \frac{\partial C_{i\ell}^{k\ell}}{\partial v} \quad \forall i = 1, 2, 3, \ldots , \]

In the Galerkin projection (GP), the matrix \( M_{ij} \) is symmetric, typically diagonally dominant, and given by the expression

\[ M_{ij}^{GP} = \left( \frac{\gamma_{ia} \gamma_{ja}}{\pi (\gamma_{ia} + \gamma_{ja})} \right)^{3/2} \exp \left[ - \frac{\gamma_{ia} \gamma_{ja}}{\gamma_{ia} + \gamma_{ja}} (\mathbf{v}_a - \mathbf{v}_a)^2 \right], \]

whereas in the center-collocation (CC) method, the matrix \( M_{ij}^{CC} \) is no longer necessarily symmetric, but can still be dominated by the diagonal components:

\[ M_{ij}^{CC} = \frac{1}{\gamma_a} \gamma_a \gamma_{ia} \gamma_{ja}^{-1} \exp \left[ - \frac{\gamma_{ia} \gamma_{ja}}{\gamma_{ia} + \gamma_{ja}} (\mathbf{v}_a - \mathbf{v}_a)^2 \right]. \]

On the right-hand-side of Eq. (2), the tensor \( C_{ab}^{k\ell} \) becomes

\[ C_{ab}^{k\ell} = \int d\mathbf{v} F_{ia}(\mathbf{v}) C_{ab}^{k\ell}(\mathbf{v}), \quad (C_{ab}^{k\ell})_{CC} = C_{ab}^{k\ell}(v_a), \]

for the Galerkin and center-collocation, respectively. Obtaining the center-collocation tensor \( (C_{ab}^{k\ell})_{CC} \) is merely a matter of evaluating the RBF tensor \( C_{ab}^{k\ell}(v) \) at the collocation points. Evaluation of the tensor \( (C_{ab}^{k\ell})_{GP} \) for the Galerkin projection is somewhat more complicated, although the result may be potentially be more accurate or robust.

Nevertheless, to maintain simplicity, we focus hereafter on the center-collocation method and omit the CC subscript for brevity. In this case the RBF equations for the full non-linear system become

\[ \sum_j M_{ij}^{CC} \frac{\partial w_{ja}}{\partial t} = \sum_{b, k, \ell} w_{ja}(t) \frac{\partial w_{j\ell}(t)}{\partial v} C_{i\ell}^{k\ell} \quad \forall i = 1, 2, 3, \ldots , \]

where the operator \( \mathcal{L} \)

\[ \mathcal{L}_{ia} \equiv \frac{\partial}{\partial t} + \mathbf{v}_a \cdot \nabla + \frac{e_a}{\gamma_a} \left[ (\mathbf{w}_a - \mathbf{v}_a) \cdot \mathbf{E} + (\mathbf{w}_a - \mathbf{v}_a) \times \mathbf{B} \right] \]

retains the familiar appearance of the Vlasov operator even though the velocity-space has been completely removed from the problem. Note that \( \mathcal{L} \) depends explicitly on species \( a \) and implicitly on \( b \) via the Maxwell equations. In \( \mathcal{L} \) we have defined the temperature-like parameter \( \gamma_a^2 = \sigma_a/2 \gamma_a \) and also dropped some additional terms that arise if the parameters \( \gamma_a \) and \( \mathbf{v}_a \) depend on position. The RBF-kinetic equation (3) describes collisional fluid-like evolution of the weights in time and space. One physically appealing feature of the RBF expansion is that familiar expressions for macroscopic fluid moments retain their intuitive form:

\[
\text{Density} \quad n_a = \sum_i w_{ia} \\
\text{Velocity} \quad n_a \mathbf{v}_a = \sum_i w_{ia} \mathbf{v}_a \\
\text{Temperature} \quad \frac{3}{2} n_a T_a = \sum_i \frac{w_{i a}}{2} \left[ 3 \sigma_a^2 + \frac{1}{2} m_a (\mathbf{v}_a - \mathbf{v}_a)^2 \right] \\
\text{Momentum flux tensor} \quad \Pi_a = \sum_i w_{i a} m_a \sigma_a^2 \mathbf{v}_a + \mathbf{v}_a \mathbf{v}_a \\
\text{Energy flux} \quad \mathcal{Q}_a = \sum_i w_{ia} \left[ \frac{5}{2} \sigma_a^2 + \frac{1}{2} m_a (\mathbf{v}_a^2) \right].
\]
Another is that also the Maxwell’s equations can be written compactly:
\[ \nabla \cdot \mathbf{E} = (1 / \varepsilon_0) \sum_a e_a \rho_a = (1 / \varepsilon_0) \sum_a e_a \phi_a, \]
\[ \nabla \times \mathbf{B} = \mu_0 \sum_a e_a \mathbf{v}_a = \mu_0 \sum a, i e_a w_a \mathbf{v}_a. \]

where for simplicity we have neglected the displacement current. In contrast to the standard fluid approach, where calculation of the higher-order velocity-space moments becomes increasingly cumbersome, the RBF-kinetic approach offers a straightforward and intuitive alternative by describing the velocity space with Gaussian functions that naturally appear in the kinetic theory as thermodynamic equilibrium solutions. Moreover, so long as the RBF spacing is equal to about one e-folding length, the problem remains well-conditioned even for hundreds or thousands of basis functions.

It should also be evident that the Gaussian RBF method is well-suited to linearized models. The linearized collision operator in this case is just the sum of \( C^0 \) and \( C^{10} \); namely, the first row and column of the non-linear collision tensor. Because the method is mesh-free, it is also naturally suited to multi-species plasmas, unlike contemporary algorithms that are based on a velocity mesh [6].

4. Non-linear simulations in 2D and 3D

To assess the viability of the Gaussian RBF approach, we study a single species plasma using a uniform grid for the collocation points \( \mathbf{v}_0 \), and a fixed value \( \gamma_2 = \gamma \) for all RBFs. We also normalize time with the collision time, i.e., \( \tau = L_0 / t \). Our problem is thus to solve the non-linear single species Fokker–Planck equation

\[ \frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial \mathbf{v}^2} \]  
\[ \frac{\partial^2 f}{\partial \mathbf{v}^2} = C[f] \tag{4} \]

We emphasize that this is a highly nontrivial numerical problem. First, we test how well the discretization preserves a (stationary) Maxwellian equilibrium state, where the temperature of the equilibrium is fixed and much warmer than the individual RBFs. For the analytic equilibrium we choose \( f_M = (\beta / \pi)^{3/2} \exp \left( -\beta |\mathbf{v}|^2 \right) \), with \( \beta = 1/4 \) and arrange the collocation points into uniform rectilinear grids in spaces \( [v_x, v_y, v_z] \in [-6.5, 6.5] \times [-6.5, 6.5] \times [-6.5, 6.5] \) and \( [v_1, v_2] \in [-6.5, 6.5] \times [0, 6.5] \) for the full 3D and axisymmetric 2D methods respectively. The equilibrium is projected onto the RBF basis to obtain the numerical equivalent of the steady state distribution function. Then the collision operator in Eq. (4) is evaluated for an increasing number of collocation points and different values of \( \gamma \). As an error metric, we choose max\( |C[f] / f| \), presented in Fig. 1. This test is sensitive to the error both in the region interior to the RBF collocation centers, as well as to the region beyond them. As is evident from Fig. 1, the maximum of the global value decreases when the number of collocation points is increased indicating that a numerical equivalent to the analytical steady-state can be reached.

Next, we follow the relaxation of a non-trivial distribution function to the equilibrium state and track the conservation of number, momentum, and energy densities. The initial state of the distribution function is set to \( f(v, \tau = 0) = \sum_{\mu = 1}^{6} \frac{1}{\sqrt{2\pi}} \exp \left( -[v-w]^2 / 2 \right) \), where \( \mu = 1/5, v_1 = (3, 0, 0), v_2 = (-3, 0, 0) \), for the \( (v_x, v_y, v_z) \) components respectively. The initial state is thus axisymmetric with respect to the axis \( v_3 \), and the axisymmetric setting is thus chosen to be \( (v_1 = v_3, v_2 = v_3 + v_2) \). The collocation points are chosen uniformly in the regions \( [v_x, v_y, v_z] \in [-8, 8] \times [-8, 8] \times [-8, 8] \) and \( [v_1, v_2] \in [-12, 12] \times [0, 12] \). The initial state projected to the Gaussian RBF basis to get initial values for the weights, and the weights then evolve in time according to Eq. (2) using a standard fourth-order Runge-Kutta method. From the time-evolution of the weights, we have calculated the evolution of the velocity-space moments, and recorded their maximal deviations from the initial values during the simulation. The results are illustrated in Fig. 2, where one observes good conservation of number (\( n \)), momentum (velocity components \( v_x, v_y, \) and \( v_z \)), and energy (\( \mathcal{E} \)) densities for both full 3D, and axisymmetric 2D solvers.

Even better conservation properties can be reached by embedding the conservation laws into the time-evolution of the weights. For example, replacing one row in the matrix \((M^2)_{\mathbf{c}}\) with ones (unity moments of the basis functions) and setting the corresponding element in \((C^{hi})_{\mathbf{c}}\) to zero neglects information from one of the basis nodes but improves density conservation. With this approach we observed, e.g., density conservation up to 12 digits without relevant increase in the computation time.

The time evolution of the distribution in the conservation study using 15\(^3\) 3D RBFs and 30 \times 15 axisymmetric RBFs is illustrated in Fig. 3 with slices in \((v_x, v_y)\)-plane for the full 3D solver and in \((v_x, v_y)\)-plane for the 2D axisymmetric solver. The time slices are chosen for the initial state \( (\tau = 0) \) in Figs. 3(a) and 3(e), for the beginning of the relaxation process \( (\tau = 3) \) in Figs. 3(b) and 3(f), for the merging phase towards the equilibrium \( (\tau = 6) \) in Figs. 3(c) and 3(g), and for the final state \( (\tau = 40) \) where the distribution function is close to spherical symmetry and equilibrium in Figs. 3(d) and 3(h). Both the 2D and 3D solvers qualitatively describe the same solution to the initially axisymmetric problem and preserve the initial symmetry. As a more quantitative measure, we also follow the time evolution of the \( v^4 \)-moment of the distribution function which is not a conserved quantity. From the
analytical initial state one can calculate the value to be $4.98 \times 10^4$ and for the corresponding analytical equilibrium state the value is $5.65 \times 10^4$. The time evolution of $v^4$-moment from the numerical calculation is shown in Fig. 4. As is clear, the 2D and 3D solutions agree with each other and also confirm our claim that the solution in Figs. 3(d) and 3(h) is close to the analytical equilibrium state.

5. Discussion

The tests described above show that the RBF approach is able to accurately treat non-linear collisional relaxation in both 2D and 3D velocity space. Although a thorough comparison and benchmark against other solvers is outside the scope of the present study, we note that the conservation properties of the new method are at least comparable to the approach described in Ref. [2]. In particular, for non-linear axisymmetric relaxation in 2D, the new scheme gives a relative error in the energy moment of roughly $10^{-4}$ for $30 \times 15$ velocity space RBFs. In Fig. 4 of Ref. [2], where a similar relaxation study starting with a slightly less non-linear initial state is shown, a relative error of $10^{-3}$ is found for $128 \times 128$ grid points.

We have also observed that conservation error is sensitive to both the size of the domain covered by the RBF collocation points and the choice of $\gamma \Delta^2$ (inverse flatness of the RBFs). First, reducing the domain the collocation points cover introduces a residual error because a larger portion of the velocity space is neglected. This affects conservation properties as illustrated in Fig. 5 for density. Second, the choice of $\gamma \Delta^2$ affects the condition number $\kappa(M)$ of the RBF collocation matrix, as illustrated in Fig. 6. The condition number then affects the accuracy of the initial projection to the RBF basis and the time evolution since Eq. (3) is multiplied by the inverse of the RBF matrix during every explicit Runge–Kutta step.

The condition number issue is well known in the context of RBF interpolation (see Ref. [7] and references therein). Although increasing flatness leads to interpolants which are more accurate, it also leads to large condition number of the RBF matrix and corresponding loss of accuracy. In our tests, we observed the condition number and the residual error to have a combined effect. Expanding the collocation domain while keeping the absolute adjacent
distance of the RBF collocation points equal improves the conservation properties: In Fig. 5 the yellow curve at $N = 15$ and the green curve at $N = 11$ both correspond to $\Delta = 1$ but the density conservation is better by two orders of magnitude with the larger domain. Carrying out similar sensitivity test for the collocation domain as in Fig. 5 but for different $\gamma \Delta^2$ showed overall degradation of the conservation properties for smaller $\gamma \Delta^2$, as one would expect due to increase in the condition number.

It is not surprising that the residual error is important. It is a common source of error in methods that have to deal with infinite domains: spectral methods may suffer from aliasing effects due to periodic basis functions (see Ref. [6] and references therein) and finite difference or element methods face the problem of setting boundary conditions (see Ref. [2]).

6. Summary

In this Letter, we have derived a completely new approach to collisional dynamics in plasmas. We have shown how to implement a Gaussian radial-basis-function ansatz to discretize the velocity space, and demonstrated its capabilities in non-linear Fokker–Planck calculations in both full 3D and axisymmetric 2D velocity-space. In a wider connection to kinetic theory we have provided also the discretization of the advection terms and given analytical ansatz expressions for all relevant fluid-moments.

In the future, our method could be used to solve various problems where non-equilibrium kinetic effects are important, as long as the Landau-approximation for the collision integral is valid. For example, time-evolution of velocity space loss-cones and holes, often encountered in pedestal physics in magnetic fusion experiments, could be of interest. In addition, the method could be applied to describe how fast particle populations slow-down and deposit energy and momenta to main plasma. We also expect the method to find applications in linear problems.

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